AN INEQUALITY FOR CLOSED SPACE CURVES

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1. Among a number of interesting results in a paper of I. Fáry (see [2]) appears the following. Let C be a rectifiable closed curve of length $L(C)$ and total curvature $\kappa(C)$ enclosed by a sphere S of radius r in Euclidean 3-space. Then

$$L(C) \leqq \frac{4}{\pi} r\kappa(C) . \tag{1}$$

The proof of (1) rests upon the corresponding inequality for plane closed curves, which states that if C is enclosed by a circle of radius r, then

$$L(C) \leqq r\kappa(C) . \tag{2}$$

The latter inequality gives a sharp result, with equality obtained in case C is a circle of radius r.

In this paper we sharpen (1) to the following result. Let C be a rectifiable closed curve enclosed by a $k-1$ dimensional sphere S of radius r in Euclidean k-space, $k \geqq 2$. Then

$$L(C) \leqq r\kappa(C) . \tag{3}$$

The proof of (3) again depends on the plane case and is motivated by the following construction. We form the cone T over the curve C with apex at the center of S, slit along a longest generator and develop the result in a plane. The resulting plane arc C' is completed to a closed plane curve C'' by attaching an arc of a circle. It is noted that the curvature of C' is equal pointwise to the geodesic curvature of C with respect to T, which in turn is not greater, pointwise, than the curvature of C. The length of C' is the same as that of C. The inequality (2) applied to C'' now gives (3).

2. In this section we prove some lemmas which lead directly to the main theorem.

LEMMA 1. *Let C be a rectifiable plane arc of length L. For any line G, let $n(p, \theta)$ be the number of intersections of G with C, where (p, θ), $p \geqq 0$, $0 \leqq \theta < 2\pi$, are the normal coordinates of G. Then*

$$L = \tfrac{1}{2}\int_0^{2\pi}\int_0^{\infty} n(p, \theta)\, dp d\theta . \tag{4}$$

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This striking formula of Crofton is proved by Blaschke, [1], page 46.

LEMMA 2. *Let C be a closed plane curve parametrized by arc length s. Let $\vec{r} = \vec{r}(s)$, $0 \leqq s \leqq L$, be the tracing vector of, C, and assume $\vec{r}''$ exists and is continuous except at a finite number of points $\vec{r}(s_1), \cdots, \vec{r}(s_m)$, where there are corners with "exterior" angles $\alpha_1, \cdots, \alpha_m$ respectively. Given any direction θ, $0 \leqq \theta < 2\pi$, let $n(\theta)$ be the number of tangents to C orthogonal to that direction, where a tangent to C at $\vec{r}(s_i)$, $i = 1, 2, \cdots, m$, means a line through the point but not crossing C at that point. Then*

$$(5) \qquad \frac{1}{2}\int_0^{2\pi} n(\theta)\, d\theta = \int |\vec{r}''(s)|\, ds + \sum_{i=1}^{m} \alpha_i = \textit{total curvature of } C\,,$$

where the integral on the right is extended over the smooth part of C.

Proof. We may write $n(\theta) = \sum_{i=0}^{m} n_i(\theta)$, where $n_0(\theta)$ counts the number of tangents to the smooth part of C and $n_i(\theta)$, $i \neq 0$, counts the number of tangents at $\vec{r}(s_i)$. Clearly n_i takes only the values 0 or 1, for $i \neq 0$, and

$$(6) \qquad \frac{1}{2}\int_0^{2\pi} n_i(\theta)\, d\theta = \alpha_i,\ i \neq 0\ .$$

Finally, we have that

$$(7) \qquad \frac{1}{2}\int_0^{2\pi} n_0(\theta)\, d\theta = \int |\vec{r}''(s)|\, ds\ ,$$

since the left hand side is just the measure of the spherical image (counting multiplicity) of the smooth part of C.

LEMMA 3. *Let $\vec{x}_0, \vec{x}_1, \cdots, \vec{x}_n$, be the successive vertices of a plane polygon $\bar{P}$ enclosed by a circle S of radius r_0. Suppose further that the "initial" and "end" points, $\vec{x}_0$ and $\vec{x}_n$ respectively, lie on S. Let α_i, $0 \leqq \alpha_i \leqq \pi$, be the angle between $\vec{x}_{i+1} - \vec{x}_i$ and $\vec{x}_i - \vec{x}_{i-1}$, $i = 1, \cdots, n-1$. If $\vec{x}_0 \neq \vec{x}_n$, let α_0, $0 \leqq \alpha_0 \leqq \pi$, be the angle between $\vec{x}_1 - \vec{x}_0$ and the unit tangent vector to S (with counterclockwise orientation) at $\vec{x}_0$, and let α_n, $0 \leqq \alpha_n \leqq \pi$, be the angle between $\vec{x}_n - \vec{x}_{n-1}$ and the unit tangent vector to S (with counterclockwise orientation) at $\vec{x}_n$. If $\vec{x}_0 = \vec{x}_n$, then simply let $\alpha_0 (= \alpha_n)$, $0 \leqq \alpha_0 \leqq \pi$, be the angle between $\vec{x}_1 - \vec{x}_0$ and $\vec{x}_0 - \vec{x}_{n-1}$. Let $L(\bar{P})$ be the length of $\bar{P}$.*

Then if $\vec{x}_0 \neq \vec{x}_n$, we have that

$$(8) \qquad L(\bar{P}) \leqq r_0 \sum_{i=0}^{n} \alpha_i\ .$$

If $\vec{x}_0 = \vec{x}_n$, we have

$$L(\bar{P}) \leqq r_0 \sum_{i=0}^{n-1} \alpha_i \ . \tag{8'}$$

(This lemma is a special case of Fáry's theorem for the plane. See [2], page 121. The proof we give here is essentially that of Fáry.)

Proof. We consider first the case where $\vec{x}_0 \neq \vec{x}_n$. Let $\bar{S}$ be the arc of S traversed in a counterclockwise direction in going along S from $\vec{x}_n$ to $\vec{x}_0$. Let $C = \bar{P} \cup \bar{S}$. Let δ be the angle subtended at the center of S by $\bar{S}$. Then Lemma 1 gives,

$$L(\bar{P}) + r_0\delta = L(C) = \tfrac{1}{2}\int_0^{2\pi}\int_0^{r_0} n(p, \theta)\ dpd\theta \ . \tag{9}$$

It is easy to see, however, that $n(p, \theta) \leqq n(\theta)$ for $0 \leqq \theta < 2\pi$. Hence, by (9) and (5), we have

$$L(\bar{P}) + r_0\delta \leqq \tfrac{1}{2}r_0\int_0^{2\pi} n(\theta)\, d\theta = r_0\Bigl(\sum_{i=0}^{n} \alpha_i + \delta\Bigr) \ . \tag{10}$$

This gives the assertion for $\vec{x}_0 \neq \vec{x}_n$. The case $\vec{x}_0 = \vec{x}_n$ is now clear.

LEMMA 4. *Let P be a closed polygon enclosed by a $k-1$ dimensional sphere S of radius r in Euclidean k-space. Let $\vec{y}_0, \vec{y}_1, \cdots, \vec{y}_n = \vec{y}_0$, be the successive vertices of P. Let β_i, $0 \leqq \beta_i \leqq \pi$, be the angle between $\vec{y}_{i+1} - \vec{y}_i$ and $\vec{y}_i - \vec{y}_{i-1}$, $i = 0, 1, \cdots, n-1$, where $\vec{y}_{-1}$ is defined to be $\vec{y}_{n-1}$. Define the total curvature, $\kappa(P)$, of P, by*

$$\kappa(P) = \sum_{i=0}^{n-1} \beta_i \ , \quad \text{(\textit{See Milnor}, [3], p. 249.)} \tag{11}$$

Let $L(P)$ be the length of P. Then

$$L(P) \leqq r\ \kappa(P) \ . \tag{12}$$

Proof. Let $\vec{o}$ be the center of S. Assume that the vertices of P are labeled so that $\vec{y}_0$ is no closer to $\vec{o}$ than any other vertex. Let β_i', $0 \leqq \beta_i' \leqq \pi$, be the angle between $\vec{y}_i - \vec{o}$ and $\vec{y}_i - \vec{y}_{i+1}$; let β_i'', $0 \leqq \beta_i'' \leqq \pi$, be the angle between $\vec{y}_i - \vec{o}$ and $\vec{y}_i - \vec{y}_{i-1}$, $i = 0, 1, \cdots, n-1$. The triangle inequality applied to a spherical triangle cut out of a sphere centered at $\vec{y}_i$ shows that

$$\beta_i' + \beta_i'' \geqq \pi - \beta_i, \text{ and } (\pi - \beta_i') + (\pi - \beta_i'') \geqq \pi - \beta_i \ .$$

Hence,

$$|\pi - (\beta_i' + \beta_i'')| \leqq \beta_i \ , \quad i = 0, 1, \cdots, n-1 \ . \tag{13}$$

We now form the cone over P with apex at $\vec{o}$, cut along the edge connecting $\vec{o}$ to $\vec{y}_0$ and develop the result in a plane as follows. Let $\vec{p}$ be a fixed point in the plane R^2. We map $\vec{y}_0$ into any point $\vec{x}_0 \in R^2$ satisfying $|\vec{x}_0 - \vec{p}| = |\vec{y}_0 - \vec{o}| = r_0$. We next map $\vec{y}_1$ into a point $\vec{x}_1 \in R^2$ satisfying $|\vec{x}_1 - \vec{p}| = |\vec{y}_1 - \vec{o}| = r_1$, and such that the angle δ_1, from $\vec{x}_0 - \vec{p}$ to $\vec{x}_1 - \vec{p}$, measured in a counterclockwise direction, is equal to the angle δ_1, $0 \leqq \delta_1 \leqq \pi$, between $\vec{y}_0 - \vec{o}$ and $\vec{y}_1 - \vec{o}$. In general we map $\vec{y}_i$ into $\vec{x}_i \in R^2$ so that $|\vec{x}_i - \vec{p}| = |\vec{y}_i - \vec{o}| = r_i$ and the angle δ_i from $\vec{x}_{i-1} - \vec{p}$ to $\vec{x}_i - \vec{p}$, measured counterclockwise, is equal to the angle δ_i, $0 \leqq \delta_i \leqq \pi$, between $\vec{y}_{i-1} - \vec{o}$ and $\vec{y}_i - \vec{o}$. This construction gives us a polygon $\bar{P}$ in R^2. Construct the circle S' of radius r_0 centered at $\vec{p}$. Then $\bar{P}$ is enclosed by S', and $\vec{x}_0$ and $\vec{x}_n$ (in general $\vec{x}_0 \neq \vec{x}_n$) are on S'. It is easily seen that the angle α_i, $0 \leqq \alpha_i \leqq \pi$, between $\vec{x}_i - \vec{x}_{i-1}$ and $\vec{x}_{i+1} - \vec{x}_i$, is $|\pi - (\beta'_i + \beta''_i)|$, $i = 1, 2, \cdots, n-1$. It is also seen that the angles α_0 and α_n described in Lemma 3 are equal to $(\pi/2) - \beta'_0 > 0$ and $(\pi/2) - \beta''_0 > 0$ respectively if $\vec{x}_0 \neq \vec{x}_n$ and are both equal to $\pi - (\beta'_0 + \beta''_0) > 0$ if $\vec{x}_0 = \vec{x}_n$. Hence if $\vec{x}_0 \neq \vec{x}_n$,

$$\begin{aligned}(14) \qquad \sum_{i=0}^{n} \alpha_i &= \frac{\pi}{2} - \beta'_0 + \sum_{i=1}^{n-1} |\pi - (\beta'_i + \beta''_i)| + \frac{\pi}{2} - \beta''_0 \\ &= \sum_{i=0}^{n-1} |\pi - (\beta'_i + \beta''_i)| ,\end{aligned}$$

and if $\vec{x}_0 = \vec{x}_n$,

$$(14') \qquad \sum_{i=0}^{n-1} \alpha_i = \sum_{i=0}^{n-1} |\pi - (\beta'_i + \beta''_i)| .$$

Therefore, by (8), (8′), (14), and (14′),

$$L(P) = L(\bar{P}) \leqq r_0 \sum_{i=0}^{n-1} |\pi - (\beta'_i + \beta''_i)| \leqq r_0 \sum_{i=0}^{n-1} \beta_i = r_0 \kappa(P) \leqq r\kappa(P) .$$

3. THEOREM 1. *Let C be a rectifiable closed curve enclosed by a $k-1$ dimensional sphere S of radius r in Euclidean k-space, $k \geqq 2$. Let $L(C)$ be the length of C and $\kappa(C)$ be the total curvature of C. ($\kappa(C) =$ l.u.b. $\kappa(P)$, where P runs over all polygons inscribed in C. See Milnor,* [3].) *Then*

$$L(C) \leqq r\kappa(C) .$$

Proof. Given any $\varepsilon > 0$, there is a polygon P inscribed in C such that $L(C) - L(P) \leqq \varepsilon$. We have that $\kappa(P) \leqq \kappa(C)$. Hence

$$L(C) - \varepsilon \leqq L(P) \leqq r\kappa(P) \leqq r\kappa(C) .$$

The theorem follows.

COROLLARY. *Let C be a closed curve of class C'' enclosed by a unit $k-1$ dimensional sphere in Euclidean k-space. Let $\kappa(s) = |\vec{r}''(s)| =$ curvature of C at $\vec{r}(s)$, $0 \leqq s \leqq L(C)$. Then*

$$\max \kappa \geqq 1 . \tag{15}$$

Proof.

$$L(C) \leqq \kappa(C) = \int_0^{L(C)} \kappa(s)\, ds \leqq \max \kappa \cdot L(C) .$$

Note that we have used the fact that the above integral form for the total curvature coincides with the previous definition. This is proved by Milnor in [3].

REFERENCES

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