HAUSDORFF DIMENSION OF LEVEL SETS OF SOME RADEMACHER SERIES

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1. Introduction. A special case of a result of Kaczmarz and Steinhaus [4] (Theorem 2 with $a = b$) shows that if $\{a_i\}$ $(i = 1, 2, \cdots)$ is a sequence of real numbers with $\sum_{i=1}^{\infty} |a_i| = +\infty$ and $a_i \to 0$, then the Rademacher series $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes every preassigned real value c (cardinal number of the continuum) times for x in $(0, 1]$. One object of this paper is to refine this result in certain directions. We shall prove

THEOREM 1. *If the sequence $\{a_i\}$ is in l_2, but not in l_1, then $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes every preassigned real value on a set of Hausdorff dimension* 1.

We shall also prove

THEOREM 2. *If $\{a_i\}$ is a sequence of bounded variation $(\sum_{i=1}^{\infty} |a_i - a_{i-1}| < \infty)$ which is not in l_1 but $a_i \to 0$, then $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes each preassigned real value on a set of Hausdorff dimension at least* 1/2.

In § 6, we apply the method of proof to a problem on the distribution of digits in decimal expansions of numbers.

In § 7 through 11, we develop a theory of dimension of level sets for series of the type $\sum_{i=1}^{\infty} r^i R_i(x)$ where r is a fixed number in the interval $[1/2, 1)$.

2. Preliminary definitions and lemmas.

DEFINITION 1. The i^{th} $(i = 1, 2, \cdots)$ Rademacher function is defined to be $R_i(x) = 1 - 2\varepsilon_i(x)$ $(0 < x \leqq 1)$, where $\varepsilon_i(x)$ is the i^{th} digit of the (unique) nonterminating binary expansion of x.

DEFINITION 2. Let X be a subset of Euclidean n-space. Let $J_\varepsilon(X)$ be a finite or countably infinite set of open spheres $\{J_i\}$ $(i = 1, 2, \cdots)$ with finite diameters $|J_i|$ whose union covers X and whose diameters do not exceed ε where $\varepsilon > 0$. The Hausdorff outer measure of order s,

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where s is a positive number, is defined as

$$\Lambda^s X = \lim_{\varepsilon \to 0} \inf_{J_\varepsilon(X)} \sum |J|^s ,$$

where the summation is extended over the members of $J_\varepsilon(X)$ and $\inf_{J_\varepsilon(X)}$ is with respect to all admissible $J_\varepsilon(X)$. The Hausdorff dimension of X is defined as $\dim X = \inf_{s \geqq 0} \{s \mid \Lambda^s X = 0\}$.

DEFINITION 3. Suppose $x \in (0, 1]$. $T_n(x)$ shall be the point in Euclidean n-space whose jth coordinate is given by $T_n^j(x) = \sum_{i=0}^{\infty} \varepsilon_{in+j}(x)\, 2^{-(i+1)}$ $(j = 1, 2, \cdots, n)$.

LEMMA 1. *If x is a binary irrational (not of the form $p/2^k$ with p and k integers), then $R_{(i-1)n+j}(x) = R_i(T_n^j(x))$ for $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, n$.*

LEMMA 2. *If A is a subset of $(0, 1]$, then n dim A = dim $T_n(A)$.*

Proof. A binary cube (or binary interval in the case of the line) is defined as a closed cube in n-dimensional space whose 2^n vertices are of the form

$$\left(\frac{k_1 + \delta_1}{2^m}, \frac{k_2 + \delta_2}{2^m}, \cdots, \frac{k_n + \delta_n}{2^m}\right),$$

where the δ_i $(i = 1, 2, \cdots, n)$ assume independently the values 0 or 1, the k_i are nonnegative integers less than 2^m, and m is a positive integer. The cube is denoted by $W_{k_1, k_2, \cdots, k_n; m}$ or W. For $n = 1$, I is written in place of W. It can be shown that an equivalent definition of dimension is obtained (for a subset of the unit cube $[0 \leqq x_i \leqq 1\ (i = 1, 2, \cdots, n)]$ in n-space) if one replaces in Definition 2 the spheres by binary cubes and uses the cube edge $(1/2^m)$ in place of the sphere diameter.

Let $k_i/2^m = \sum_{j=1}^{m} \varepsilon_j^i 2^{-j}$, where ε_j is 0 or 1. With the cube $W_{k_1, k_2, \cdots, k_n; m}$ we associate the closed interval I:

$$\left[\sum_{j=1}^{m}\sum_{i=1}^{n} \varepsilon_j^i\, 2^{-[(j-1)n+i]}, \sum_{j=1}^{m}\sum_{i=1}^{n} \varepsilon_j^i\, 2^{-[(j-1)n+i]} + 2^{-mn}\right]$$

and write $I = s(W)$. Let $\{I^n\}$ denote the set of all binary intervals on [0,1] of length of the form 2^{-kn} $(k = 0, 1, 2, \cdots)$. s is a one-to-one mapping between $\{I^n\}$ and the set of all binary cubes in n-space, and hence has an inverse s^{-1}. We note that $l(s(W)) = (e((W))^n$ where l denotes the length of the interval and e denotes length of the cube edge

We show first that dim $T_n(A) \leqq n$ dim A. It suffices to assume that A contains no points of the form $p/2^k$ (p and k are integers), for if it did, they could be deleted without changing the dimension of A or $T_n(A)$.

Suppose positive δ and ε are given arbitrarily. There exists a covering of A by binary intervals I_i $(i = 1, 2, \cdots)$ such that $\sum_{i=1}^{\infty} (l(I_i))^{\dim A+\delta} < \varepsilon$. Here we make use of analogue of Theorem 16.1 of [8] for Hausdorff measures defined using binary cube coverings. Replace the covering I_i $(i = 1, 2, \cdots)$ by a covering of intervals from $\{I^n\}$ by replacing each interval of I_i of length 2^{-p_i} by $2^{p_i^*-p_i}$ intervals of length $2^{-p_i^*}$ where p_i^* is the smallest integer greater than p_i which is a multiple of n. Denote the resulting covering intervals by I_i^n $(i = 1, 2, \cdots)$. The cubes $s^{-1}(I_i^n)$ will cover $T_n(A)$ and

$$\sum_{i=1}^{\infty} (e(s^{-1}(I_i^n)))^{n(\dim A+\delta)} = \sum_{i=1}^{\infty} (l(I_i^n))^{\dim A+\delta} < 2^n\varepsilon .$$

Since this holds for each pair of positive ε and δ, $\dim T_n(A) \leqq n \dim A$.

We now show that $\dim A \leqq 1/n \dim T_n(A)$. Suppose that positive ε and δ are given arbitrarily. There exists a covering of $T_n(A)$ by binary cubes W^i $(i = 1, 2, \cdots)$ such that $\sum_{i=1}^{\infty} (e(W_i))^{\dim T_n(A)+\delta} < \varepsilon$. Let W_j^i $(j = 1, 2, \cdots, k(i))$ $(k(i) \leqq 3^n)$ be the binary cubes of edge $e(W^i)$ which intersect the cube W^i, including W^i itself. The closed binary intervals $s(W_j^i)$ $(j = 1, 2, \cdots, k(i); i = 1, 2, \cdots)$ cover A and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{k(i)} (l(s(W_j^i)))^{(1/n)(\dim T_n(A)+\delta)} \leqq 3^n \sum_{i=1}^{\infty} (e(W^i))^{\dim T_n(A)+\delta} < 3^n\varepsilon .$$

Thus $\dim A \leqq (1/n) \dim T_n(A)$. This completes the proof of Lemma 2.

We remark that Lemma 2 has an analogue for the well-known Peano curve (see [3], pages 457–8) which maps the unit interval into an n-dimensional cube.

LEMMA 3. *If $\{a_i\}$ is in l_2, then $\sum_{i=1}^{\infty} a_i R_i(x)$ converges almost everywhere.*

This lemma is due to Rademacher. See Theorem 3 in [4].

LEMMA 4. *If $\sum_{i=1}^{\infty} |a_i| = \infty$ and $a_i \to 0$, then given any real number α, there exists a binary irrational $x_0 \in (0,1)$ such that $\sum_{i=1}^{\infty} a_i R_i(x_0) = \alpha$.*

See Theorem 2 in [4]. The proof of this lemma is similar to that of Riemann's theorem that any conditionally convergent series of real numbers can be rearranged to converge to any preassigned real number.

LEMMA 5. *If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ are convergent series of real numbers, then $\sum_{i=1}^{\infty} a_i + b_i$ is convergent with value $\sum a_i + \sum b_i$.*

This is Theorem 3, page 78, of [6].

DEFINITION 4. *A subset A of* $(0,1]$ *is of type* $G(n, K, M)$ (n, K, M *nonnegative integers*) *if it has the following property. Suppose* ε'_{ni+K} ($i = M, M+1, \cdots$) *is an arbitrarily given sequence of* 0's *and* 1's. *Then there exists* $x \in A$ *such that* $\varepsilon_{ni+K}(x) = \varepsilon'_{ni+K}$ ($i = M, M+1, \cdots$).

LEMMA 6. *If* $A \subset (0,1]$ *is of type* $G(n, K, M)$, *then* $\dim A \geqq 1/n$.

LEMMA 6A. *If* $A \subset (0,1]$ *is simultaneously of types* $G(2n+1,1,0)$, $G(2n+1,3,0), \cdots, G(2n+1, 2n-1,0)$ *where* n *is a positive integer, then* $\dim A \geqq n/(2n+1)$.

Lemmas 6 and 6A follow from Lemma 2.

3. Proof of Theorem 1. Let α be a real number. Let n be an integer > 1. Let

$$E = \left\{(x_1, x_2, \cdots, x_n) \,\middle|\, \sum_{j=1}^{n} \sum_{i=1}^{\infty} R_i(x_j) a_{(i-1)n+j} = \alpha \right\} ,$$

where $0 < x_j \leqq 1$. Let E' be the subset of E whose points have only binary irrational coordinates. Let $x' = (x'_1, x'_2, \cdots, x'_n)$ be in E' and let x be the (unique) inverse image of x' under T_n; i.e., $x' = T_n(x)$. Observe that x is a binary irrational number. We have

$$\begin{aligned} \alpha &= \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(x'_j) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(T_n^j(x)) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_{(i-1)n+j}(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{(i-1)n+j} R_{(i-1)n+j}(x) \qquad (1) \\ &= \sum_{i=1}^{\infty} a_i R_i(x) . \end{aligned}$$

Lemmas 1 and 5 justify the third and fourth equalities, respectively. Let $\beta(\alpha, \{a_i\})$ be the set of all x in $(0,1]$ such that $\alpha = \sum_{i=1}^{\infty} a_i R_i(x)$. It follows from (1) that the x defined above is in $\beta(\alpha, \{a_i\})$ and hence that $T_n(\beta) \supset E'$. We now show that $\dim E' \geqq n-1$. For some integer $j (1 \leqq j \leqq n)$, we must have $\sum_{i=1}^{\infty} |a_{(i-1)n+j}| = \infty$. Without loss of generality, we take $j = n$. Let A_j ($j = 1, 2, \cdots, n-1$) be the subset of the interval $(0 < x_j \leqq 1]$ where $\sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(x_j)$ converges and A'_j be the subset of A_j whose points are binary irrational. Let $A^* = X_{1 \leqq j \leqq n-1} A'_j$ be the Cartesian product of the A'_j. Suppose $x^* \in A^*$ and $x^* = (x^*_1, x^*_2, \cdots, x^*_{n-1})$. Suppose $\sum_{j=1}^{n-1} \sum_{i=1}^{\infty} R_i(x^*_j) a_{(i-1)n+j} = \alpha_1$. By Lemma 4, there exists a binary irrational number x^*_n such that $\sum_{i=1}^{\infty} a_{(i-1)n+n} R_i(x^*_n) = \alpha - \alpha_1$. Thus $\sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{(i-1)n+j} R_i(x^*_j) = \alpha_1 + \alpha - \alpha_1 = \alpha$. Hence $(x^*_1, x^*_2, \cdots, x^*_n) \in E'$. By Lemma 3, the measure of A_j and A'_j is 1. Thus $\dim A^* = n-1$. But since the projection of E' on the $X_{1 \leqq j \leqq n-1} x_j$ hyperplane includes A^*, $\dim E' \geqq n-1$.

Using Lemma 2, we have $\dim \beta(\alpha, \{a_i\}) = (1/n) \dim T_n(\beta(\alpha, \{a_i\})) \geqq (1/n) \dim E' \geqq (1/n)(n-1) = 1 - (1/n)$. Since this holds for every integer $n > 1$, the theorem follows.

4. Proof of Theorem 2. Let n be an integer > 1. We can assume, without loss of generality, that the a_i are positive. Since $\{a_i\}$ is not in l_1, at least one of the $2n+1$ sequences $\{a_{(2n+1)i-2n}\}, \{a_{(2n+1)i-(2n-1)}\}, \cdots, \{a_{(2n+1)i}\}$ $(i = 1, 2, \cdots)$ is not in l_1. We suppose, without loss of generality, that $\{a_{(2n+1)i}\}$ is not in l_1. We take $s_i = \pm 1$. Choose $s_{(2n+1)i-(2n-j)}$ $(j = 0, 2, 4, \cdots, 2n-2; i = 1, 2, \cdots)$ as an arbitrary sequence of $+1$'s and -1's except that an infinity are -1. Put $s_{(2n+1)i-(2n-j-1)} = -s_{(2n+1)i-(2n-j)}$. The series $\sum_{i=1}^{\infty} s_{(2n+1)i-(2n-j)}\, a_{(2n+1)i-(2n-j)} + s_{(2n+1)i-(2n-j-1)}\, a_{(2n+1)i-(2n-j-1)}$ converges since $\sum (a_i - a_{i-1})$ converges absolutely and hence any subseries $\sum' (a_i - a_{i-1})$ converges absolutely. Call its value α_{2n-j}.

Now let α be the preassigned value and let $\alpha' = \alpha - \sum_{j=0}^{2n-2} \alpha_{2n-j}$. Choose, by Lemma 4, $s_{(2n+1)i} = \pm 1$ so that $\sum s_{(2n+1)i}\, a_{(2n+1)i} = \alpha'$. With these choices for s_i, $\sum_{i=1}^{\infty} s_i a_i = \alpha$. Remembering that $\varepsilon_i(x) = (1 - R_i(x))/2$, from Lemma 6A, we have that the set on which $\sum a_i R_i(x) = \alpha$ has dimension at least $n/(2n+1)$. Since n is an arbitrary integer > 1, the theorem follows.

5. Remarks.

1. Theorem 1 could be slightly improved as follows. We could consider the sets $\beta(\gamma, \delta, \{a_i\})$ of x where for preassigned numbers γ and δ $(-\infty \leqq \gamma \leqq \delta \leqq +\infty)$, $\overline{\lim}_{n\to\infty} \sum_{i=1}^{\infty} a_i R_i(x) = \delta$ and $\underline{\lim}_{n\to\infty} \sum_{i=1}^{\infty} a_i R_i(x) = \gamma$. If a_i is in l_2 but not in l_1, then $\dim \beta(\gamma, \delta, \{a_i\}) = 1$.
2. It might be interesting to investigate the measure of $\beta(\alpha, \{a_i\})$ under the hypothesis of Theorem 1. It might also be interesting to determine, if possible, the dimension function (dimension in sense of [2]) of $\beta(\alpha, \{a_i\})$.
3. The conclusion of Theorem 2 is not as precise as that of Theorem 1. However, it may be the best possible conclusion.
4. The function sequence $\{a_i R_i(x)\}$ is a probablistically independent function sequence. No explicit use of this property is made, but we believe that this property is implicitly used. We hope later to consider extensions to other probablistically independent function sequences; also extensions to certain lacunary trigonometric series should be considered. We note that $R_k(x) = \text{sign}\, \{\sin 2^k \pi x\}$.

6. Application. Using the method of proof of Theorem 1, we prove

THEOREM 3. *Let* $\bar{L}(x) = \overline{\lim}_{n\to\infty} (1/n) \sum_{i=1}^{n} \varepsilon_i(x)$ *and* $\underline{L}(x) =$

$\underline{\lim}_{n\to\infty}(1/n)\sum_{i=1}^{n}\varepsilon_i(x)$, *where* $x\in(0,1]$. *Let* $B=\{x\mid \bar{L}x>\underline{L}x\}$. *Then* $\dim B=1$.

Proof. We shall show that $\dim\{x\mid \bar{L}(x)-\underline{L}(x)\geqq(1/k)\}\geqq(k-1)/k$, where k is an integer >1. Let $B_j=\{x_j\mid \bar{L}(x_j)=\underline{L}(x_j)=1/2; 0<x_j<1\}$ $(1\leqq j\leqq k-1)$ and E^* be the Cartesian product of all the B_j. The linear measure of B_j is 1. Now fix an $(x_1, x_2, \cdots, x_{k-1})$ in E^*. Let $E^{*\prime}$ be the subset of E^* whose points have binary irrational coordinates. Choose x_k in (0,1) such that $\bar{L}(x_k)=1$ and $\underline{L}(x_k)=0$. For example, x_k could be the decimal (base 2)

$$\cdot\underbrace{11}_{2^{2^0}}\underbrace{0\ 0\ 0\ 0}_{2^{2^1}}\cdots\underbrace{11\cdots1}_{2^{2^n}}\underbrace{0\ 0\cdots0}_{2^{2^{n+1}}}\cdots.$$

Let E be the subset of the unit cube $(x_1, x_2, \cdots, x_k)$ $(0\leqq x_j\leqq 1, 1\leqq j\leqq k-1)$ such that $(x_1, x_2, \cdots, x_{k-1})\in E^*$ and x_k as chosen above. Obviously, $\dim E=k-1$. We have, for $x\in A=\{x\mid T_n(x)\in E\}$,

$$\begin{aligned}\frac{1}{nk}&\sum_{i=1}^{nk}\varepsilon_i(x)\\ &=\frac{1}{k}\left\{\sum_{p=1}^{k-1}\frac{1}{n}\sum_{j=0}^{n-1}\varepsilon_{jk+p}(x)+\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{jk}(x)\right\}\\ &=\frac{1}{k}\left\{\sum_{p=1}^{k-1}\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j(x_p)+\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j(x_k)\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{1}{nk}\sum_{i=1}^{nk}\varepsilon_i(x)&=\frac{1}{k}\left\{\left(\sum_{p=1}^{k-1}\bar{L}(x_p)\right)+\bar{L}(x_k)\right\}\\ &=\frac{1}{k}\left\{\sum_{p=1}^{k-1}\frac{1}{2}+1\right\}=\frac{1}{k}\left\{(k-1)\frac{1}{2}+1\right\}=\frac{1}{2}+\frac{1}{2k}.\end{aligned}$$

Similarly,

$$\underline{\lim_{n\to\infty}}\frac{1}{nk}\sum_{i=1}^{nk}\varepsilon_i(x)=\frac{1}{2}-\frac{1}{2k}.$$

Since $B\supset\{x\mid \bar{L}(x)-\underline{L}(x)\geqq 1/k\}$, we have, using Lemma 2, $\dim B\geqq\dim A=1/k\dim E=(1/k)(k-1)=1-1/k$ for every integer $k>1$. The theorem follows.

7. Geometric series. In § 7–11, we investigate the Hausdorff dimension of the set

$$\beta(\alpha, \{r^i\})=\{x\mid \sum_{i=1}^{\infty}r^iR_i(x)=\alpha; 0<x\leqq 1\},$$

where r is a fixed number in $(1/2,1)$, and $-\sum_{i=1}^{\infty} r^i < \alpha < \sum_{i=1}^{\infty} r^i$. The sets β are closed, but not necessarily perfect. Since $\sum_{i=1}^{\infty} r^i R_i(x)$ converges absolutely, it is sufficient to consider the sets $\beta_1(\alpha,r) = \{x \mid \sum_{i=1}^{\infty} r^i \varepsilon_i(x) = \alpha; 0 < x \leqq 1\}$ with $0 < \alpha < \sum_{i=1}^{\infty} r^i$. Since the dimension of a set is not changed by adding to it a countable set, we add to $\beta_1(\alpha,r)$ those binary rationals $p/2^k$ (p,k are nonnegative integers) for which $\sum_{i=1}^{\infty} r^i \varepsilon_i'(p/2^k) = \alpha \, (0 \leqq p/2^k \leqq 1)$, where $\varepsilon_i'(p/2^k)$ is the i^{th} digit of the finite binary expansion of $p/2^k$. For the remainder of this paper we take $\alpha \in (0, \sum_{i=1}^{\infty} r^i)$. We take $\log \equiv \log_2$.

8. Preliminary lemmas.

DEFINITION 5. For $x \in (0,1]$ and $2^{-1/(n-1)} < r \leqq 2^{-1/n}$ with n a fixed integer > 1, we define $T_{n,r}(x)$ as the point in n-space whose jth coordinate is given by

$$T_{n,r}^j(x) = (1 - r_n) \sum_{i=0}^{\infty} \varepsilon_{in+j}(x)\, r^{ni} \qquad (j = 1, 2, \cdots, n)\,. \tag{2}$$

If x is of the form $p/2^k$ (p,k integers), $T_{n,r}^j(x)$ shall be two valued; one value is given by (2) and the other value is given by (2) with $\varepsilon_{in+j}(x)$ replaced by $\varepsilon_{in+j}'(x)$ arising from the finite binary expansion. In addition, $T_{n,r}^j(0) = 0$ $(j = 1, 2, \cdots, n)$.

LEMMA 7. *If A is a subset of* $[0,1]$, *then* $\dim A = |\log r| \dim T_{n,r}(A)$. The proof is similar to that of Lemma 2.

DEFINITION 6. Suppose $r \leqq 2^{-1/n}$. C_r^1 is the Cantor set of constant dissection constructed as follows. Divide the closed interval $[0,1]$ into three intervals by the points $r^n, 1 - r^n$ and remove the open middle interval of length $1 - 2r^n$. Repeat this process on the remaining left and right intervals, removing middle intervals of length $(1 - 2r^n)r^n$. The process is continued indefinitely. The set remaining is C_r^1. C_r^n is the Cartesian product of C_r^1 with itself n times.

DEFINITION 7. If $2^{-1/(n-1)} < r \leqq 2^{-1/n}$, $l_{r,\alpha}$ is the n-space hyperplane: $\sum_{i=1}^{n} r^i x_i = \alpha(1 - r^n)$.

LEMMA 8. $T_{n,r}([0,1]) \equiv C_r^n$.

Proof. Suppose $z \in T_{n,r}([0,1])$. The coordinates of z are given by expressions of type $(1 - r^n) \sum \varepsilon_i r^{ni}$ and hence are in C_r^n (see [9[). Thus, $T_{n,r}([0,1]) \subset C_r^n$. Now suppose $z \in C_r^n$ and $z = (x_1, x_2, \cdots, x_n)$. There exist $\varepsilon_i^j = 0$ or 1 such that $x_j = (1 - r^n) \sum_{i=1}^{\infty} \varepsilon_i^j r^{ni}$ $(j = 1, 2, \cdots, n)$. Let $x = \sum_{i=1}^{\infty} \sum_{j=1}^{n} \varepsilon_i^j 2^{-(ni+j)}$. At least one of the values of $T_{n,r}(x)$ is z. Thus,

$z \in T_{n,r}([0,1])$ and hence $C_r^n \subset T_{n,r}([0,1])$.

LEMMA 9. *Except possibly for a countable set,* $T_{n,r}(\beta_1(\alpha,r)) \equiv l_{r,\alpha} \cap C_r^n$.

Proof. Let $z \in l_{r,\alpha} \cap C_s^n$ and $z = (x_1, x_2, \cdots, x_n)$.

Since $z \in l_{r,\alpha}$, $\sum_{i=1}^n r^i x_i = \alpha(1 - r^n)$. Since $z \in C_r^n$, $x_i = (1 - r^n) \sum_{j=1}^\infty \varepsilon_j^i \, r^{n(j-1)}$ $(i = 1, 2, \cdots, n)$ with $\varepsilon_j^i = 0$ or 1. Possibly only a finite number of the ε_j^i are different from zero (see [9]). Choose $x = \sum_{j=1}^\infty \sum_{i=1}^n \varepsilon_j^i \, (1/2)^{n(j-1)+i}$.

If all the ε's but a finite number are zero, then (a) $\varepsilon_j^i = \varepsilon'_{n(j-1)+i}(x)$. Otherwise, we have (b) $\varepsilon_j^i = \varepsilon_{n(j-1)+i}(x)$. In case (b),

$$\sum_{i=1}^\infty \varepsilon_i(x) r^i = \sum_{j=1}^\infty \sum_{i=1}^n \varepsilon_{n(j-1)+1}(x) r^{n(j-1)+i} = \sum_{j=1}^\infty \sum_{i=1}^n \varepsilon_j^i r^{n(j-1)+i}$$
$$= \sum_{i=1}^n \sum_{j=1}^\infty \varepsilon_j^i r^{n(j-1)+i} = \sum_{i=1}^n r^i \sum_{j=1}^\infty \varepsilon_j^i r^{n(j-1)} = \sum_{i=1}^n r^i \frac{x_i}{1 - r^n}$$
$$= \frac{1}{1 - r^n} \sum_{i=1}^n r^i x_i = \frac{1}{1 - r^n} \alpha(1 - r^n) = \alpha \, .$$

A similar computation holds in case (a) with $\varepsilon_i(x)$ replaced by $\varepsilon_i'(x)$. Hence $z \in \beta_1(\alpha,r)$. Also for this x, one of the values of $T_{n,r}(x)$ is z. At most a countable number of x can have two values for $T_{n,r}(x)$. Hence, except for a countable number, $z = T_{n,r}(x) \in T_{n,r}(\beta_1(\gamma,r))$. Therefore, except for a countable set, $l_{r,\alpha} \cap C_r^n \subseteqq T_{n,r}(\beta_1(\alpha,r))$. Similar work shows that except for at most a countable number of values of $T_{n,r}(\beta_1)$, $T_{n,r}(\beta_1(\alpha,r)) \subseteqq l_{r,\alpha} \cap C_r^n$.

LEMMA 10. $\dim C_r^n = 1/|\log r|$.

This follows from Theorem 5 of [2].

9. Case where r is a root of 2. We consider the case $r = 2^{-1/n}$ with n an integer > 1 and obtain

THEOREM 4. $\dim \beta_1(\alpha,2^{-1/n}) = 1 - 1/n$.

Proof. Suppose $n > 1$. In this case, C_r^n is the unit cube and $\dim (l_{2^{-1/n},\alpha} \cap C_r^n) = n - 1$. Using Lemmas 7 and 9, we have

$$\dim \beta_1(\alpha,2^{-1/n}) = |\log 2^{-1/n}| \dim T_{n,r}(\beta_1) = \frac{1}{n} \dim (l_{2^{1/n},\alpha} \cap C_r^n)$$
$$= 1 - 1/n \, .$$

If $n = 1$, $\sum \varepsilon_i(x) r^i = \sum \varepsilon_i(x) 2^{-i}$ assumes every value on $(0,1]$ exactly once.

10. Bounds on dimension of β_1.

THEOREM 5. *For $1 > r \geqq 1/2$ and for almost every α,* $\dim \beta_1(\alpha,r) \leqq \log 2r$.
If $2^{-(1/n)} \geqq r \geqq 2^{-1/(n-1)}$ *for* $n > 1$, *then* $\dim \beta_1(\alpha,r) \leqq 1 - 1/n$ *for every* α.

Proof. From Lemma 7, $\dim T_{n,r}([0,1]) = 1/|\log r|$. Marstrand's theorem [7], when generalized to n dimensions, states that, for almost every α, the hyperplane $l_{r,\alpha}$ intersects the set $T_{n,r}([0,1])$ in a set of dimension $\leqq 1/|\log r| - 1$. Thus, from Lemmas 7 and 9, $\dim(\beta_1(\alpha,r)) = |\log r| \dim T_{n,r}(\beta_1(\alpha,r)) = |\log r| \dim (l_{r,\alpha} \cap C_r^n) \leqq |\log r|\,(1/|\log r| - 1) = \log 2r$.

We now prove the second part. We need to show that the dimension of $l_{\alpha,r} \cap C_r^n$ is less than or equal to the dimension of C_r^{n-1}. Roughly, we proceed as follows. C_r^{n-1} is a perfect set constructed in Cantor fashion from nested cubes which we call W_j^{n-1} $(j = 1, 2, \cdots)$. These are the cubes of edge r^{nm_j} (m_j a positive integer) which are the $(n-1)$-dimensional Cartesian products of the closed non-middle intervals used in constructing C_r^1. We denote by W_j^n the corresponding n-dimensional cube whose base is W_j^{n-1}. We show that it requires at most 2^n "translates" of each W_j^n to cover $l_{r,\alpha} \cap C_r^n$.

For arbitrary positive ε and ε, there exists a subsequence of the W_j^{n-1} such that

$$\sum{}' \,|\, W_j^{n-1} |^{\dim C_r^{n-1} + \varepsilon} < \delta \;,$$

where Σ' indicates summation over a subsequence of $j = 1, 2, \cdots$, and $|\, W_j^{n-1}|$ denotes cube edge. Consider one of the cubes W_j^{n-1} $(j = 1, 2, \cdots)$, say W_l^{n-1}. Let

$$((k_1^l + \delta_1)r^{nm_l}, (k_2^l + \delta_2)r^{nm_l}, \cdots, (k_{n-1}^l + \delta_{n-1})r^{nm_l})$$

be the 2^{n-1} vertices of $W_{l^{n-1}}$. Here the δ_i assume independently the values 0 or 1, and the k_i^l are certain integers.

The x_n coordinates of the intersections of the lines in n-space $x_i = (k_i^l + \delta_i)r^{nm_j}$ $(i = 1, 2, \cdots, n-1)$ with the hyperplane $l_{r,\alpha}$ are given by

$$x_n = \Big[\alpha(1 - r^n) - \sum_{i=1}^{n-1} r^i(k_i + \delta_i)r^{nm_l}\Big]\Big/r^n \;.$$

The extreme values of these intersections are

$$x_n^0 = \Big[\alpha(1 - r^n) - \sum_{i=1}^{n-1} r^i k_i r^{nm_j}\Big]\Big/r^n$$

and

$$x_n^1 = \left[\alpha(1 - r^n) - \sum_{i=1}^{n} r^i(k_i + 1)r^{nm_j}\right]\Big/ r^n .$$

We have

$$x_n^0 - x_n^1 = \frac{1}{r^n}\sum_{i=1}^{n-1} r^i r^{nm_l} \leqq r^{nm_l}/r^n(1 - r) \leqq r^{nm_l}2/(1 - 2^{-1/n}) .$$

Let $g(n)$ be three more than the largest integer in $2/(1 - 2^{-1/n})$.

Since $| W_l^{n-1} | = r^{nm_l}$, to each W_l^{n-1} there corresponds a set of at most $g(n)$ n-dimensional cubes of side r^{nm_l}, say $W_{l,1}^n, W_{l,2}^n, \cdots, W_{l,g_n}^n$ such that $\bigcup_{l=1}^{\infty} \bigcup_{p=1}^{g(n)} W_{l,p}^n$ covers $l_{\alpha,r} \cap C_r^n$ and

$$\sum_{l=1}^{\infty}\sum_{p=1}^{g(n)} | W_{l,p}^n |^{\dim C_r^{n-1}+\varepsilon} = g(n)\sum_{l=1}^{\infty} | W_l^{n-1} |^{\dim C_r^{n-1}+\varepsilon} \leqq g(n)\delta .$$

Hence, $\dim l_{\alpha,r} \cap C_r^n \leqq \dim C_r^{n-1} = (n-1)/| \log r^n |$.

Thus, using Lemma 7, we have

$$\dim \beta_1(\alpha,r) \leqq |\log r| \dim(l_{r,\alpha} \cap C_r^n) \leqq 1 - 1/n .$$

We shall show that there are members of the exceptional set of α in Theorem 5. Take $r = (\sqrt{5} - 1)/2$ and $\alpha = \sum_{i=0}^{\infty} r^{3i+1} = r/(1 - r^3)$. Note that for this r, $r = r^2 + r^3$. Now let A be all those x in $(0,1]$ for which either $\varepsilon_{3i+1}(x) = 0, \varepsilon_{3i+2}(x) = \varepsilon_{3i+3}(x) = 1$ or $\varepsilon_{3i+1}(x) = 1, \varepsilon_{3i+2}(x) = \varepsilon_{3i+3}(x) = 0$ independently for $i = 1, 2, \cdots$. Then A is of type $G(3, 1,0)$ in the sense of Definition 4. For any $x \in A$, $\sum_{i=1}^{\infty} \varepsilon_i(x)\, ((\sqrt{5} - 1)/2)^i = \alpha$ and $\dim A \geqq 1/3$. But $\log_2 (\sqrt{5} - 1) \doteqdot .31$. We remark that if $r = (\sqrt{5} - 1)/2$ and $\alpha = \sum_{k=0}^{\infty} r^{4k+1}$, then it can be shown that $\dim \beta(\alpha,r) \geqq (1/4) \log_2 (3 + \sqrt{5})/2 \doteqdot .35$.

11. Additional theorem.

THEOREM 6. *Let* $(\sqrt{5} - 1)/2 < r < 1$. *Then* $\dim \beta_1(\alpha,r) \geqq 1/n$ *where* n *is the least integer* n_0 *such that* $n_0 > [\log(2r - 1) - \log(r^2 + r - 1)]/(-\log r)$.

Note that as $r \to (\sqrt{5} - 1)/2+$, $n \to \infty$.

To prove the theorem, we need two lemmas.

LEMMA 11. *If a monotone decreasing sequence* $\{a_i\}$ *of positive terms has the property that* $a_i \leqq \sum_{j=i+1}^{\infty} a_j < +\infty$ *for all* i, *then every* $\alpha (0 < \alpha \leqq \sum_{i=1}^{\infty} a_i)$ *can be expressed as* $\alpha = \sum'^{\infty}_{i=1} a_i$, *where* Σ' *indicates some of the terms possibly are omitted from the sum. Further,* Σ' *can be required to have an infinite number of terms.*

The first sentence of the lemma is stated essentially in [5, page 547], except that there the case of $\Sigma\ (\pm)a_i$ is discussed. But one can write

$2\Sigma(^{+}_{0})a_i = \Sigma a_i + \Sigma(^{+}_{-})a_i$. The second sentence of the lemma should be obvious.

LEMMA 12. *If* $(\sqrt{5}-1)/2 < r < 1$ *and* n *is a positive integer such that* $r^n < (r^2+r-1)/(2r-1)$, *then* $\sum''^{\infty}_{i=1} r_i$ [*where* Σ'' *indicates that the terms of the form* r_{ni+1} $(i = 0, 1, 2, \cdots)$ *are omitted*] *has the property that* $r^i < \sum''^{\infty}_{j=i+1} r^j$.

Proof. $\sum''^{\infty}_{j=i+1} r^j = \sum^{\infty}_{j=i+1} r^j - \Sigma_v r^{nv+1}$ where Σ_v is over all v such that $nv + 1 \geqq i + 1$. Let v_0 be the smallest v allowed. Then

$$\sum_{j=i+1}^{\infty}{}' r^j = \frac{r^{i+1}}{1-r} - \frac{r^{nv_0+1}}{1-r^n} \geqq \frac{r^{i+1}}{1-r} - \frac{r^{i+1}}{1-r^n}$$

$$= r^i\left(\frac{r}{1-r} - \frac{r}{1-r^n}\right).$$

Therefore $\sum''^{\infty}_{j=i+1} r^j > r^i$ if $r/(1-r) - r/(1-r^n) > 1$, and hence if $r/(1-r^n) < (2r-1)/(1-r)$ which reduces to $r^n < (r^2+r-1)/(2r-1)$.

We now prove Theorem 6. Let

$$\alpha_1 = r^1 + r^{n+1} + r^{2n+1} + \cdots,$$
$$\alpha_2 = r^2 + r^3 + \cdots + r^n + r^{n+2} + \cdots + r^{2n} + r^{2n+2} + \cdots.$$

Case I. Suppose $0 < \alpha < \alpha_2$. Let

$$f(x_1) = \sum_{i=1}^{\infty} \varepsilon_i(x_1) r^{(i-1)n+1}.$$

Then

$$sup_{0<x_1\leqq 1} f(x_1) = f(1) = \alpha_1.$$

Let $\omega = \{x_1 \mid f(x_1) < \alpha\}$. Since $\lim_{x_1\to 0+} f(x_1) = 0$, ω contains an open binary interval $\omega^* = (0,(1/2^q))$ where q is an integer. Choose $x_1^* \in \omega^*$ and let $\alpha^* = f(x_1^*)$. Note that $\alpha - \alpha^* < \alpha < \alpha_2$. By Lemma 12 and then Lemma 11, there exist $\varepsilon'_k = 0, 1$ $(k = 2, 3, \cdots, n, n+2, \cdots, 2n, 2n+2, \cdots)$ (for infinitely many k, $\varepsilon'_k = 1$) such that

$$\varepsilon'_2 r^2 + \cdots + \varepsilon'_n r^n + \varepsilon'_{n+2} r^{n+2} + \cdots + \varepsilon'_{2n} r^{2n} + \cdots = \alpha - \alpha^*.$$

Choose $x = \sum^{\infty}_{i=1} \varepsilon^*_i 2^{-i}$, where $\varepsilon^*_{(i-1)n+1} = \varepsilon_i(x_1^*)$ $(i = 1, 2, 3, \cdots)$, and $\varepsilon^*_i = \varepsilon'_i$ $(i = 2, 3, \cdots, n, n+2, \cdots, 2n, 2n+2, \cdots)$. Then $\sum^{\infty}_{i=1} \varepsilon^*_i r^i = \alpha^* + \alpha - \alpha^* = \alpha$. Thus, it is possible to choose $\varepsilon^*_{(i-1)n+1}$ $(i > q)$ independently (except that infinitely many are 1) so that $\sum^{\infty}_{i=1} \varepsilon^*_i r^i = \alpha$. Hence $\beta_1(\alpha, r)$ includes a set A of type $G(n, 1, q)$ which, by Lemma 6, has dimension $\geqq 1/n$.

Case II. Suppose $\alpha_2 \leqq \alpha$. Let

$$\omega = \{x_1 | \alpha - \alpha_2 < f(x_1)\} .$$

Since $\alpha < \alpha_1 + \alpha_2$, $\alpha - \alpha_2 < \alpha_1$. Also $\lim_{x_1 \to 1^-} f(x_1) = \alpha_1$. Therefore, ω contains an open binary interval $\omega_1^* = ((2^q - 1)/2^q, 1)$. Choose $x_1^* \in \omega_1^*$ and let $\alpha^* = f(x_1^*)$. Then $\alpha - \alpha^* < \alpha - (\alpha - \alpha_2) = \alpha_2$. The proof is then completed as in Case I with ω_1^* in place in ω^*.

We remark that Theorem 6 can be generalized to other absolutely convergent Rademacher series $\sum_{i=1}^{\infty} a_i R_i(x)$; namely, those which satisfy conditions of the form $0 < a_i / \Sigma_{j>i} a_j < (\sqrt{5 - 4/n} - 1)/2$ for a fixed integer $n > 1$ and $\{a_i\}$ $(i = 1, 2, \cdots)$ a positive monotone sequence.

THEOREM 7. *If* $r \geqq 2^{-1/n}$, *then* $\dim \beta_1(\alpha, r) \geqq 1 - 1/n$.

The details of the proof will not be given since it is similar to that of Theorem 6. Since $r^n \geqq 1/2$, given α, there exists M such that $\varepsilon_{in+j} = 0, 1$ $(1 \leqq j \leqq n - 1, i > M)$ can be chosen independently and then ε_{in+j} $[(1 \leqq i \leqq M, 1 \leqq j \leqq n - 1)$ and $(j = n, i = 1, 2, \cdots)]$ determined so that $\alpha = \sum_{i=0}^{\infty} \sum_{j=1}^{n} \varepsilon_{in+j} r^{in+j}$.

Added in Proof. A sequel to this paper will appear in Proc. Amer. Math. Soc.

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