SEMICHARACTERS OF THE CARTESIAN PRODUCT OF TWO SEMIGROUPS

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1. If S and T are semigroups, then by $S \times T$ we mean the semigroup consisting of the Cartesian product $S \times T$ of the sets S and T with coordinatewise multiplication. The semigroup $S \times T$ is called the *Cartesian product of the seimgroups* S and T. A complex-valued multiplicative function on a semigroup S is called a *semicharacter* of S if it is different from 0 at some point and is bounded (1.3, [1]). The set of all semicharacters of S is denoted by $\hat{S}$.

We show that $\widehat{S \times T} = \{\chi | \chi(x, u) = \phi(x)\psi(u)$ for some $\phi \in \hat{S}$, $\psi \in \hat{T}\}$ (2.4). We obtain a similar result for continuous semicharacters of topological semigroups (3.3). One of the most interesting consequences of the above results is a theorem on prime ideals (2.6). A subset I of a semigroup S is called a *prime ideal* of S if I is a proper (i.e., $\neq S$) two-sided ideal of S whose complement in S is a semigroup. For convenience we also call the empty set a prime ideal (cf. Definitions 2, 2a, [2]). We also prove a theorem concerning continuity of the semicharacters of the Cartesian product $S \times T$ of two topological semigroups (3.4).

If A and B are sets, then $A - B$ will denote the set of all elements of A which are not contained in B. A semigroup will always be nonempty. A nonempty subset I of S is said to be an (two-sided) *ideal* of S if $xy, yx \in I$ for all $x \in S, y \in I$.

All results in this paper are stated for the Cartesian product of two semigroups. However, a simple inductive argument shows that all of them generalize to the Cartesian product of any finite number of semigroups.

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2. If S and T are semigroups with two-sided identities, then semicharacters of $S \times T$ are obtained easily from the semicharacters of S and T. (If e and f are identities of S and T, respectively, then each element (x, u) of $S \times T$ can be written as $(x, f)(e, u)$.) In 5, [3], Št. Schwarz considers this case for commutative semigroups. We first introduce two definitions.

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2.1. DEFINITION. Let f and g be arbitrary complex-valued functions defined on sets S and T, respectively. We define the function (f, g) on $S \times T$ by $(f, g)(x, u) = f(x)g(u)$ for all $x \in S, u \in T$.

2.2. DEFINITION. Let S and T be semigroups. We define $\hat{S} \circ \hat{T} = \{\chi | \chi = (\phi, \psi)$ for some $\phi \in \hat{S}, \psi \in \hat{T}\}$.

2.3. THEOREM. *Let S and T be semigroups and let $\chi \in \widehat{S \times T}$. Then χ can be written uniquely as (ϕ, ψ), where $\phi \in \hat{S}$ and $\psi \in \hat{T}$. If (a, b) is any element of $S \times T$ such that $\chi(a, b) \neq 0$, then*

$$\phi(x) = \frac{\chi(ax, b)}{\chi(a, b)} \text{ for all } x \in S \text{ and}$$

$$\psi(u) = \frac{\chi(a, bu)}{\chi(a, b)} \text{ for all } u \in T.$$

Proof. Let (a, b) be any element of $S \times T$ such that $\chi(a, b) \neq 0$ and let x and y be elements of S. Then $\chi(ax, b)\chi(a, b) = \chi(axa, b^2) = \chi(a, b)\chi(xa, b)$ and after dividing this identity by $\chi(a, b)$, we obtain

$$(1) \qquad \chi(ax, b) = \chi(xa, b) \text{ for all } x \in S .$$

Let

$$\phi(x) = \frac{\chi(ax, b)}{\chi(a, b)} \text{ for all } x \in S .$$

From (1) we obtain

$$\chi(ax, b)\chi(ay, b) = \chi(ax, b)\chi(ya, b) = \chi(axya, b^2) = \chi(axy, b)\chi(a, b)$$

and consequently

$$\phi(x)\phi(y) = \frac{\chi(ax, b)}{\chi(a, b)} \frac{\chi(ay, b)}{\chi(a, b)} = \frac{\chi(axy, b)\chi(a, b)}{\chi(a, b)\chi(a, b)} = \phi(xy) \text{ for all } x, y \in S.$$

We let

$$\psi(u) = \frac{\chi(a, bu)}{\chi(a, b)} \text{ for all } u \in T .$$

Like ϕ, ψ is multiplicative. Let (x, u) be any element of $S \times T$. By (1), we have

$$\begin{aligned}\chi(ax, b)\chi(a, bu) &= \chi(a, bu)\chi(ax, b) = \chi(a, bu)\chi(xa, b) = \chi(axa, bub) \\ &= \chi(a, b)\chi(x, u)\chi(a, b)\end{aligned}$$

and thus

$$\phi(x)\psi(u) = \frac{\chi(ax, b)}{\chi(a, b)} \frac{\chi(a, bu)}{\chi(a, b)} = \frac{\chi(a, b)\chi(x, u)\chi(a, b)}{\chi(a, b)\chi(a, b)} = \chi(x, u) .$$

Therefore

$$\chi = (\phi, \psi) . \tag{2}$$

Since $\chi(a, b)$ is a constant, ϕ is bounded, and since $\phi(a) \neq 0$, we conclude that $\phi \in \hat{S}$. A similar argument shows that $\psi \in \hat{T}$.

It only remains to prove uniqueness of ϕ and ψ. Suppose now that $(\phi, \psi) = (\phi_1, \psi_1)$. Then $\phi(x)\psi(u) = \phi_1(x)\psi_1(u)$ for all $x \in S$, $u \in T$. There exists an element $u_0 \in T$ such that $\psi(u_0) \neq 0$. Hence

$$\phi(x) = \frac{\psi_1(u_0)}{\psi(u_0)}\phi_1(x) \text{ for all } x \in S .$$

Let $K = \psi_1(u_0)/\psi(u_0)$. If x_0 is an element of S such that $\phi(x_0) \neq 0$, then $\phi(x_0^2) = [\phi(x_0)]^2 = [K\phi_1(x_0)]^2 = K[K\phi_1(x_0^2)] = K\phi(x_0^2)$ and thus $K = 1$ since $\phi(x_0) \neq 0$. Therefore $\phi = \phi_1$. One shows similarly that $\psi = \psi_1$.

2.4. COROLLARY. *If S and T are semigroups, then $\widehat{S \times T} = \hat{S} \circ \hat{T}$.*

Proof. If $\phi \in \hat{S}$ and $\psi \in \hat{T}$, it is easy to show that $(\phi, \psi) \in \widehat{S \times T}$. Therefore $\widehat{S \times T} \supseteqq \hat{S} \circ \hat{T}$. The reverse inclusion follows from 2.3.

The following lemma has been proved by Št. Schwarz for several classes of semigroups (Lemma 3, [2] and Lemma 3.2, [3]).

2.5. LEMMA. *Let S be a semigroup and let $\chi \in \hat{S}$. Then the set $I = \{x \in S \mid \chi(x) = 0\}$ is a prime ideal of S. Conversely, if I is a prime ideal of S, then there exists a semicharacter $\chi \in \hat{S}$ such that*

$$I = \{x \in S \mid \chi(x) = 0\} .$$

Proof. The proof of the first statement is routine and is omitted. For the converse, let I be a prime ideal of S. Define the function χ on S by

$$\chi(x) = \begin{cases} 1 \text{ if } x \in S - I \\ 0 \text{ if } x \in I \end{cases}$$

Then $\chi \in \hat{S}$ and $I = \{x \in S \mid \chi(x) = 0\}$.

2.6. THEOREM. *Let S and T be semigroups. Then a set L is a prime ideal of $S \times T$ if and only if $L = (I \times T) \cup (S \times J)$ where I and J are prime ideals of S and T, respectively.*

Proof. Let L be a prime ideal of $S \times T$. By the second part of 2.5, there is a semicharacter $\chi \in \widehat{S \times T}$ vanishing exactly on L. From 2.4 it follows that $\chi = (\phi, \psi)$ for some $\phi \in \hat{S}$, $\psi \in \hat{T}$. Clearly $\chi(x, u) = \phi(x)\psi(u) = 0$ if and only if either $\phi(x) = 0$ or $\psi(u) = 0$. Hence $L = \{(x, u) \in S \times T \mid \chi(x, u) = 0\} = (I \times T) \cup (S \times J)$, where $I = \{x \in S \mid \phi(x) = 0\}$ and $J = \{u \in T \mid \psi(u) = 0\}$. By the first part of 2.5, I and J are prime ideals of S and T, respectively.

Conversely, let I and J be prime ideals of S an T, respectively. By the second part of 2.5, there are semicharacters $\phi \in \hat{S}$, $\psi \in \hat{T}$ vanishing exactly on I and J, respectively. From 2.4 it follows that $(\phi, \psi) = \chi$ for some $\chi \in \widehat{S \times T}$. Clearly $\chi(x, u) = \phi(x)\psi(u) = 0$ if and only if either $\phi(x) = 0$ or $\psi(u) = 0$, and this happens if and only if either $x \in I$ or $u \in J$. Thus $L = (I \times T) \cup (S \times J) = \{(x, u) \in S \times T \mid \chi(x, u) = 0\}$, and hence by the first part of 2.5, L is a prime ideal of $S \times T$.

3. We next consider continuous semicharacters of topological semigroups.

3.1. DEFINITION. A semigroup S is called a *topological semigroup* if S is also a topological space and the mapping of $S \times S$ into S defined by $(x, y) \rightarrow xy$ is a continuous mapping of $S \times S$ into S. The set of all continuous semicharacters of S will be denoted by $\hat{S}_c$.

It is straightforward to prove that if S and T are topological semigroups, then $S \times T$ is a topological semigroup under the product topology.

3.2. DEFINITION. If S and T are topological semigroups, we define $\hat{S}_c \circ \hat{T}_c = \{\chi \mid \chi = (\phi, \psi)$ for some $\phi \in \hat{S}_c, \psi \in \hat{T}_c\}$.

3.3. THEOREM. *If S and T are topological semigroups, then* $(\widehat{S \times T})_c = \hat{S}_c \circ \hat{T}_c$.

Proof. If $\phi \in \hat{S}_c$ and $\psi \in \hat{T}_c$, then $(\phi, \psi) \in \widehat{S \times T}$ by 2.4. Hence to show that $(\phi, \psi) \in (\widehat{S \times T})_c$, it suffices to show that (ϕ, ψ) is continuous in both variables at an arbitrary point of $S \times T$. Using the fact that ϕ and ψ are bounded, the proof of this fact is a standard continuity argument and is omitted. Therefore $(\widehat{S \times T})_c \supseteq \hat{S}_c \circ \hat{T}_c$. The reverse inclusion follows from 2.4 and the fact that joint continuity implies continuity in each variable.

3.4. THEOREM. *Let S and T be topological semigroups and let $\chi \in \widehat{S \times T}$. Then the following statements are true.*

(a) *Let $\phi \in \hat{S}$ be such that $(\phi, \psi) = \chi$ for some $\psi \in \hat{T}$. If there exists $(a, b) \in S \times T$ such that $\chi(a, b) \neq 0$ and $\chi(y, b)$ is a continuous function of y either in aS or in Sa, then $\phi \in \hat{S}_c$.*

(b) *$\chi(x, d)$ is continuous in S for each $d \in T$ if and only if for some $(a, b) \in S \times T$ such that $\chi(a, b) \neq 0$ and $\chi(y, b)$ is continuous either in aS or in Sa.*

(c) *$\chi \in (\widehat{S \times T})_c$ if and only if for some $(a, b) \in S \times T$ such that $\chi(a, b) \neq 0$, $\chi(y, b)$ is continuous either in aS or in Sa, and for some $(c, d) \in S \times T$ such that $\chi(c, d) \neq 0$, $\chi(c, v)$ is continuous either in dT*

or in Td.

Proof. (a) By 2.3, we have $\phi(x) = \chi(ax, b)/\chi(a, b)$ for all $x \in S$. Since (a, b) is fixed, it suffices to show that $\chi(ax, b)$ is a continuous function of x in S. Suppose that $\chi(y, b)$ is continuous in aS. Let $m(x) = ax$ for all $x \in S$ and $l(y) = \chi(y, b)$ for all $y \in aS$. Then m is continuous by continuity of multiplication and l is continuous by hypothesis. We have $l \circ m(x) = \chi(ax, b)$ for all $x \in S$. Since $l \circ m$ is continuous, $\chi(ax, b)$ is continuous in x. Hence $\phi \in \hat{S}_c$.

Suppose now that $\chi(y, b)$ is continuous in Sa. By (1) of 2.3, we have $\chi(ax, b) = \chi(xa, b)$ and consequently $\phi(x) = \chi(xa, b)/\chi(a, b)$ for all $x \in S$. Defining $m(x) = xa$ for all $x \in S$, we show that $\phi \in \hat{S}_c$ in a similar way as above.

(b) Necessity is obvious; we prove sufficiency. Let d be any element of T. If $\chi(x, d) = 0$ for all $x \in S$, then $\chi(x, d)$ is continuous in S. Suppose that $\chi(c,d) \neq 0$ for some $c \in S$. Continuity of $\chi(y, b)$ in aS or in Sa implies that $\phi \in \hat{S}_c$, where $\phi(x) = \chi(ax, b)/\chi(a, b)$ for all $x \in S$, by part (a) of the present theorem and 2.3. By 2.3, ϕ is unique and thus $\chi(ax, b)/\chi(a, b) = \chi(cx, d)/\chi(c, d)$ for all $x \in S$. Consequently, $\chi(cx, d)/\chi(c, d)$ is continuous in x. We have

$$\begin{aligned}\chi(x, d) &= \frac{\chi(c^2, d)\chi(x, d)}{\chi(c^2, d)} = \frac{\chi(c^2x, d^2)}{\chi(c^2, d)} \\ &= \frac{\chi(c, d)\chi(cx, d)}{\chi(c^2, d)} = \frac{\chi(c^2, d^2)\chi(cx, d)}{\chi(c^2, d)\chi(c, d)}\end{aligned}$$

for all $x \in S$. Since $\chi(c^2, d^2)/\chi(c^2, d)$ is a constant, $\chi(x, d)$ is continuous in S.

(c) Necessity is obvious; we prove sufficiency. By 2.3, $\chi = (\phi, \psi)$ for some $\phi \in \hat{S}$, $\psi \in \hat{T}$, and by part (a) of the present theorem, $\phi \in \hat{S}_c$ and similarly $\psi \in \hat{T}_c$. From 3.3 it follows that $\chi = (\phi, \psi) \in (\widehat{S \times T})_c$.

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