TAUBERIAN CONSTANTS FOR THE $[J, f(x)]$ TRANSFORMATIONS

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1. Introduction. Let $\{s_n\}$ $(n \geqq 0)$ $(s_n = a_0 + \cdots + a_n)$ be a sequence of real or complex numbers. Denote by $t(x)$ a linear transform T

$$t(x) = \sum_{n=0}^{\infty} c_n(x) s_n$$

of $\{s_n\}$ supposed convergent for all sufficiently large values of x. In addition to classical Abelian and Tauberian theorems which give information about one of $\lim_{x \to \infty} t(x)$ and $\lim_{n \to \infty} s_n$ when the other exists, it is possible to find estimates of

$$\limsup_{n \to \infty,\, k_n \to \infty} | t(x_n) - s_n |$$

when neither $\lim t(x)$ nor $\lim s_n$ is supposed to exist but $\{a_n\}$ is assumed to satisfy the condition

$$\limsup_{n \to \infty} | n a_n | < +\infty . \tag{1.1}$$

Such estimates were obtained first by H. Hadwiger [4] for the Abel transform $t(x)$. Delange [3] developed a general theory for such estimates when $x_n = qn$, where q is some fixed positive number. Usualy the estimates proved have the form

$$\limsup_{n \to \infty, x_n \to \infty} | t(x_n) - s_n | \leqq C. \limsup_{n \to \infty} | n a_n | .$$

We call the constant C a Tauberian constant associated with the transformation T.

In this paper we shall prove some Hadwiger-type inequalities for a class of $[J, f(x)]$ transformations (see § 2). In these results the connection between n and x_n will be more general than the relation $x_n = qn$.

As a consequence of the main result of this paper we shall obtain the interesting result that for any sequence $\{s_n\}$ satisfying (1.1) the set of limit points of $\{s_n\}$ and the set of limit points of the Borel transform of $\{s_n\}$ are the same set

2. The $[J, f(x)]$ transformations. The class of $[J, f(x)]$ transformations was defined in [5], where it was shown that the $[J, f(x)]$ transfor-

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mations are the sequence-to-function analogues to the Hausdorff transformations. The definition of the $[J, f(x)]$ transformations is the following.

DEFINITION. Suppose $f(x)$ is a real or complex function defined for all $x > x_0 \geqq 0$. Let $f^{(n)}(x)$ exists for all $x > x_0$ and $n = 1, 2, \cdots$. For a given sequence $\{s_n\}$ $(n \geqq 0)$ we define the $[J, f(x)]$ transform $t(x)$ of $\{s_n\}$ by

$$t(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} f^{(n)}(x) s_n$$

supposed convergent for all sufficiently large x. We say that $\{s_n\}$ is summable $[J, f(x)]$ to s *if* $\lim_{x \to \infty} t(x) = s$.

We shall denote in the rest of this paper by $d_n(x)$, $n = 0, 1, \cdots, x > 0$, the function $(-1)^n (x^n/n!) f^{(n)}(x)$.

EXAMPLES. The $[J, f(x) \equiv (x + 1)^{-1}]$ transformation is the Abel transformation. The $[J, f(x) \equiv e^{-x}]$ transformation is the Borel transformation. The $[J, f(x) \equiv (x + 1)^{-\alpha}](\alpha > -1)$ transformation is known as the $A^{(\alpha)}$ transformation.

The following necessary and sufficient conditions for the regularity (that is that for each convergent sequence the T transform of the sequence is also convergent to the same limit) of the $[J, f(x)]$ transformations were obtained in [5].

THEOREM (2.A). *The $[J, f(x)]$ transformation is regular if, and only if, there exists a function*

$$\beta(t) \text{ of bounded variation in } 0 \leqq t \leqq 1 \tag{2.1}$$

such that

$$\beta(0) = \beta(0+) = 0, \beta(1) = \beta(1 - 0) = 1 \tag{2.2}$$

and

$$f(x) = \int_0^1 t^x d\beta(t) \qquad \text{for} \quad x > x_0 \,. \tag{2.3}$$

3. **Tauberian constants.** One of the main results of this section is finding the best (in a certain sense) Tauberian constant associated with a certain class of $[J, f(x)]$ transformations.

THEOREM (3.1). *Suppose the $[J, f(x)]$ transformation is regular, that is (by Theorem (2.A)) there exists a function $\beta(t)$ satisfying (2.1), (2.2) and (2.3). Let $q > 0$ be a constant. If $\beta(t)$ is a nondecreasing function and the integrals*

$$(3.1)\qquad \int_0 \frac{\beta(t)}{t\log 1/t}\,dt,\ \int^{\infty-} \frac{1-(t+1)f(t)}{t(t+1)}\,dt = \lim_{x\uparrow\infty}\int^{x} \frac{1-(t+1)f(t)}{t(t+1)}\,dt$$

exist, then for any sequence $\{s_n\}$ *satisfying* (1.1) *we have*

$$(3.2)\qquad \limsup_{n\to\infty,\,x\to\infty,\,nx^{-1}\to q}\left|s_n-\sum_{m=0}^{\infty} d_m(x)s_m\right| \leqq A_q\cdot\limsup_{n\to\infty}|na_n|$$

where

$$(3.3)\qquad A_q = \gamma(\textit{Euler's constant}) + \log q - \int_0^{\infty-}\frac{1-(t+1)f(t)}{t(t+1)}\,dt + 2\int_0^{e^{-v}}\frac{\beta(t)}{t\log 1/t}\,dt\,.$$

Moreover, the constant A_q *is the best in the following sense. There is a real sequence* $\{s_n\}$ *such that* $0<\limsup_{n\to\infty}|na_n|<+\infty$ *and the members of the inequality* (3.2) *are equal.*

In the proof of Theorem (3.1) we shall use the following two theorems. The first is proved by R. P. Agnew in [1], and the second is proved in [6], Theorem 7d, page 295.

THEOREM (3.A). *Suppose* $\{s_n\}$ *is any bounded (real or complex) sequence. Let* $\{c_n(x)\}$ *be a sequence of functions defined for* $0<x<\infty$ *and satisfying*

$$(3.4)\qquad \lim_{x\to\infty} c_n(x) = 0 \qquad \textit{for}\quad n=0,1,\cdots\,.$$

$$(3.5)\qquad \limsup_{x\to\infty}\sum_{n=0}^{\infty}|c_n(x)| = M < +\infty\,.$$

Then we have

$$(3.6)\qquad \limsup_{x\to\infty}\left|\sum_{n=0}^{\infty}c_n(x)s_n\right| \leqq M\cdot\limsup_{n\to\infty}|s_n|\,.$$

Moreover, M is the best constant in the following sense. There exists a bounded sequence $\{s_n\}$, $\limsup|s_n|>0$, *such that the members of inequality* (3.6) *are equal.*

THEOREM (3.B). *If* $\beta(t)$ *is a normalized function of bounded variation in* $0\leqq t\leqq R$ *for every* $R>0$ *(that is* $\beta(0)=0$, $\beta(t)=\frac{1}{2}\{\beta(t-0)+\beta(t+0)\}$*) and if the integral*

$$(3.7)\qquad f(x) = \int_0^{\infty-} e^{-xt}\,d\beta(t)$$

converges for some x, then

$$\lim_{x\to\infty}\sum_{n=0}^{[xt]} d_n(x) = \beta(t)\ , \qquad \textit{for} \quad 0 < t < \infty\ . \tag{3.8}$$

Proof of Theorem (3.1). We have formally, for $x > 0$,

$$\begin{aligned} s_n - \sum_{m=0}^{\infty} d_m(x) s_m &= \sum_{k=0}^{n} a_k - \sum_{m=0}^{\infty} d_m(x) \sum_{k=0}^{m} a_k \\ &= \sum_{k=0}^{n} a_k - \sum_{k=0}^{\infty} a_k \sum_{m=k}^{\infty} d_m(x)\ . \end{aligned} \tag{3.9}$$

We have (see [5])

$$\sum_{m=0}^{\infty} d_m(x) = 1\ , \qquad 0 < x < \infty\ . \tag{3.10}$$

Therefore, at least formally, for $x > 0$, $n = 1, 2, \cdots$,

$$\begin{aligned} s_n - \sum_{m=0}^{\infty} d_m(x) s_m &= \sum_{k=1}^{n} (ka_k)k^{-1}\Big\{1 - \sum_{m=k}^{\infty} d_m(x)\Big\} \\ &\quad - \sum_{k=n+1}^{\infty} (ka_k)k^{-1} \sum_{m=k}^{\infty} d_m(x)\ . \end{aligned} \tag{3.11}$$

From the fact that $\beta(t)$ is nondecreasing it follows that

$$d_k(x) \geqq 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad k = 0, 1, \cdots\ . \tag{3.12}$$

In order to justify our computations for sequences $\{s_n\}$ satisfying (1.1) it is enough, by (3.12), to show that the two expressions

$$\sum_{k=1}^{n} k^{-1}\Big\{1 - \sum_{m=k}^{\infty} d_m(x)\Big\}$$

$$\sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x)$$

converge for $x > 0$. The convergence of the last two expressions will follow from (3.15), (3.20) and (3.22). Suppose the convergence of the last two expressions was proved then in order to complete the proof it is enough to show, by (3.11) and Theorem (3.A), that we have

$$\lim_{x\to\infty} k^{-1}\Big\{1 - \sum_{m=k}^{\infty} d_m(x)\Big\} = 0 \qquad \text{for } k = 1, 2, \cdots\ . \tag{3.13}$$

$$\lim_{n\to\infty,\, x\to\infty,\, nx^{-1}\to q} \Big\{\sum_{k=1}^{n} k^{-1}\Big[1 - \sum_{m=k}^{\infty} d_m(x)\Big] + \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x)\Big\} = A_q\ . \tag{3.14}$$

Now, by (3.10),

$$1 - \sum_{m=k}^{\infty} d_m(x) = \sum_{m=0}^{k-1} d_m(x)\ .$$

From

$$d_m(x) = \frac{x^m}{m!}\int_0^1 t^x \left(\log \frac{1}{t}\right)^m d\beta(t) , \qquad \text{for} \quad m = 0, 1, \cdots$$

we get $\lim_{x\to\infty} d_m(x) = 0$ for $m = 0, 1, \cdots$, which proves (3.13). The expression in brackets on the left-hand side of (3.14) might be written in the form

$$\begin{aligned}(3.15)\qquad & \sum_{k=1}^{n} k^{-1}\Big[1 - \sum_{m=k}^{\infty} d_m(x)\Big] + \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) \\ &= \sum_{k=1}^{n} k^{-1} - \sum_{k=1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) + 2 \sum_{k=n+1}^{\infty} k^{-1} \sum_{m=k}^{\infty} d_m(x) \\ &\equiv \sum_{k=1}^{n} k^{-1} - S_1(x) + 2S_{2,n}(x) .\end{aligned}$$

Now, by (3.10), for $k = 1, 2, \cdots$,

$$\begin{aligned}(3.16)\qquad \frac{d}{dx} \sum_{m=k}^{\infty} d_m(x) &= \frac{d}{dx}\Big\{1 - \sum_{m=0}^{k-1} d_m(x)\Big\} \\ &= -\frac{d}{dx} \sum_{m=0}^{k-1} d_m(x) \\ &= (-1)^k \frac{x^{k-1}}{(k-1)!} f^{(k)}(x) .\end{aligned}$$

We have also

$$\begin{aligned}(3.17)\qquad \sum_{m=k}^{\infty} d_m(x) &= \int_0^{\infty} e^{-xt} \sum_{m=k}^{\infty} \frac{(xt)^m}{m!} d\{1 - \beta(e^{-t})\} \\ &= \int_0^1 \Big\{t \sum_{m=k}^{\infty} (m!)^{-1}\Big(\log \frac{1}{t}\Big)^m\Big\} d\beta(t^{1/x}) .\end{aligned}$$

By the Helly-Bray theorem (see [6], Theorem 16.4, page 31) we have from (3.17)

$$(3.18)\qquad \lim_{x\downarrow 0} \sum_{m=k}^{\infty} d_m(x) = 0 , \qquad \text{for} \quad k = 1, 2, \cdots .$$

From (3.16) and (3.18) we have

$$\sum_{m=k}^{\infty} d_m(x) = \int_0^x (-1)^k \frac{t^{k-1}}{(k-1)!} f^{(k)}(t)dt , \qquad k = 1, 2, \cdots .$$

Therefore

$$(3.19)\qquad S_1(x) = \sum_{k=1}^{\infty} \int_0^x (-1)^k \frac{t^{k-1}}{k!} f^{(k)}(t)dt$$

(and, by Fubini's theorem and (3.1), if the last integral exists)

$$= \int_0^x t^{-1} \sum_{k=1}^{\infty} d_k(t)dt$$

(and by (3.10))

$$= \int_0^x \frac{1 - f(t)}{t} dt \, .$$

The last integral exists by Fubini's theorem, because

$$\begin{aligned}\int_0^x \frac{1 - f(t)}{t} dt &= \int_0^x t^{-1}\Big\{1 - \int_0^1 u^t d\beta(u)\Big\} dt \\ &= \int_0^x dt \int_0^1 \frac{1 - u^t}{t} d\beta(u) \\ &= \int_0^1 d\beta(u) \int_0^x \frac{1 - u^t}{t} dt\end{aligned}$$

and the last integral exists, as is easy to see. Therefore

$$S_1(x) = \log(x + 1) + \int_0^x \frac{1 - (t + 1)f(t)}{t(t + 1)} dt \, . \tag{3.20}$$

Hence

$$\lim_{n \to \infty, x \to \infty, nx^{-1} \to q} \Big\{\sum_{k=1}^n k^{-1} - S_1(x)\Big\} = \gamma + \log q - \int_0^{\infty -} \frac{1 - (t + 1)f(t)}{t(t + 1)} dt \, . \tag{3.21}$$

In the same way that we obtained (3.20) we get

$$\begin{aligned}S_{2,n}(x) &= \int_0^x t^{-1} \sum_{m=n+1}^{\infty} d_m(t) dt \\ &= \int_0^x t^{-1}\Big\{1 - \sum_{m=0}^n d_m(t)\Big\} dt \\ &= \int_0^{x/n} u^{-1}\Big\{1 - \sum_{m=0}^n d_m(nu)\Big\} du \, .\end{aligned} \tag{3.22}$$

Now we get from Theorem (3.B) (with $x = nu, t = u^{-1}, 0 < u \leqq 1$) and Lebesque's theorem on the integration of boundedly convergent series, since

$$f(x) = \int_0^{\infty} e^{-xt} d\{1 - \beta(e^{-t})\}$$

that

$$\begin{aligned}\lim_{n \to \infty, x \to \infty, nx^{-1} \to q} S_{2,n}(x) &= \int_0^{q^{-1}} u^{-1} \beta(e^{-u^{-1}}) du \\ &= \int_0^{e^{-q}} \frac{\beta(v)}{v \log 1/v} dv \, .\end{aligned} \tag{3.23}$$

(3.21) and (3.23) show that (3.14) is true. This completes the proof. Q.E.D.

Using the following expression for Euler's constant

$$\gamma = \int_0^\infty \frac{1-(t+1)e^{-t}}{t(t+1)}\,dt$$

(see Bromwich [2], page 507) we obtain for A_q of Theorem (3.1) the expression

$$A_q = \log q + \int_0^{\infty-} \frac{f(t)-e^{-t}}{t}\,dt + 2\cdot\int_0^{e^{-q}} \frac{\beta(t)}{t\log 1/t}\,dt \tag{3.24}$$

and we may state Theorem (3.1) in the equivalent form

THEOREM (3.2). *If the suppositions of Theorem* (3.1) *are satisfied then the number* A_q *of Theorem* (3.1) *has the representation* (3.24).

4. Some consequences. We shall indicate here some consequences of the results of § 3.

EXAMPLE (4.1). If we choose in Theorem (3.2) $\beta(t) = 0$, for $0 \leqq t < e^{-1}$, $\beta(t) = 1$, for $e^{-1} \leqq t \leqq 1$, and $q > 0$ is a fixed number than for each sequence $\{s_n\}$ satisfying (1.1) we have

$$\limsup_{n\to\infty,\, x\to\infty,\, nx^{-1}\to q} \left| s_n - e^{-x}\sum_{m=0}^{\infty} s_m \frac{x^m}{m!} \right| \leqq |\log q| \cdot \limsup_{n\to\infty} |na_n| \,.$$

Moreover the constant $|\log q|$ is the best possible in the sense of Theorem (3.1).

Example (4.1) is an immediate consequence of Theorem (3.2).

If we choose in Example (4.1) $q = 1$ we get

THEOREM (4.1). *For a sequence* $\{s_n\}$ *satisfying* (1.1) *we have*

$$\lim_{n\to\infty,\, x\to\infty,\, nx^{-1}\to 1} \left| s_n - e^{-x}\sum_{m=0}^{\infty} s_m \frac{x^m}{m!} \right| = 0 \,;$$

and therefore the set of limit points of $\{s_n\}$ *and the set of limit points of the Borel transform of* $\{s_n\}$ *are the same set.*

Theorem (4.1) raises the following problem: It is known that Borel summability of a sequence and the condition $\sqrt{n}\,a_n = O(1)$ imply the convergence of the sequence; now by Theorem (4.1) the stronger condition $na_n = O(1)$ implies that the set of limit points of the Borel transform and the set of limit points of the sequence are the same set. We may ask therefore if it is true in general, or for which transformations it is true, that if a Tauberian condition stronger than the appropriate Tauberian condition for the transformation is satisfied then the set of limit points of the transform and the set of limit points of the sequence are the same set.

EXAMPLE (4.2). If we choose in Theorem (3.1) $\beta(t) = t$ we obtain for the Abel transformation $A_q = \gamma + \log q + 2\int_q^\infty v^{-1}e^{-v}dv$.

The last consequence was obtained by R. P. Agnew in [1].

EXAMPLE (4.3). If we choose in Theorem (3.1)

$$\beta(t) = \frac{1}{\Gamma(\alpha+1)}\int_0^t \left(\log\frac{1}{u}\right)^\alpha du\,, \qquad \alpha > -1\,,$$

then

$$A_q = \gamma + \log q - \int_0^1 \frac{1-(1-t)^\alpha}{t}dt + \frac{2}{\Gamma(\alpha+1)}\int_q^\infty v^\alpha e^{-v}\log\frac{v}{q}dv\,.$$

(3.2) assumes, in this particular case, the form, for $y = x/(x+1)$,

$$\text{(4.1)} \qquad \limsup_{n\to\infty,\, y\to 1,\, n(1-y)\to q}\left|s_n - (1-y)^{\alpha+1}\sum_{m=0}^\infty \binom{m+\alpha}{m}s_m y^m\right| \leqq A_q \cdot \limsup |na_n|\,.$$

Example (4.2) is the particular case $\alpha = 0$ of Example (4.2).

$c_n^{(\alpha)} (\alpha > -1)$ the Cesàro transform of order α of $\{s_n\}$ (in short the (C, α) transform) is defined by

$$c_n^{(\alpha)} = \left\{\binom{n+\alpha}{n}\right\}^{-1}\sum_{m=0}^n \binom{n-m+\alpha-1}{n-m}s_m\,, \qquad n = 0, 1, \cdots\,.$$

If in Example (4.3) we replace $\{s_n\}$ by the sequence $\{c_n^{(\alpha)}\}$, the (C, α) transform of a sequence $\{c_n\}$, we obtain from (4.1) the following result.

EXAMPLE (4.4). Suppose $\alpha > -1$, $q > 0$. For a sequence $\{s_n\}$ denote by $\{c_n^{(\alpha)}\}$ and $\{a_n^{(\alpha)}\}$, respectively, the sequences of the (C, α) transform of $\{s_n\}$ and $\{na_n\}$. Then for each sequence $\{s_n\}$ with a bounded sequence $\{a_n^{(\alpha)}\}$ we have

$$\text{(4.3)} \qquad \limsup_{n\to\infty,\, y\to 1,\, n(1-y)\to q}\left|c_n^{(\alpha)} - \sum_{m=0}^\infty a_m y^m\right| \leqq B_q \cdot \limsup_{m\to\infty} |a_n^{(\alpha)}|$$

where B_q is the constant A_q of Example (4.3). Moreover the constant B_q is the best in the following sense. There is a real sequence $\{s_n\}$ such that $0 < \lim \text{such}\, |a_n^{(\alpha)}| < \infty$ and the members of inequality (4.3) are equal.

5. The minimum of the function A_q. Now we investigate the behaviour of A_q of Theorem (3.1) as a function of $q > 0$.

THEOREM (5.1). *Suppose $\beta(t)$ is a function satisfying (2.1) and (2.2). Define $f(x)$ by (2.3). If $\beta(t)$ is nondecreasing and the integrals (3.1) exist then for $A_q (q > 0)$ defined by (3.3), as a function of q we* *have*

(5.1) $A_q \geqq 0$ *for* $q > 0$.

(5.2) A_q *is a continuous function for* $q > 0$.

(5.3) $\lim_{q \uparrow \infty} A_q = +\infty$.

(5.4) $\lim_{q \downarrow 0} A_q = +\infty$.

(5.5) A_q *has an absolute minimum for* $q > 0$.

(5.6) *If* $\beta(t)$ *is a continuous function the value of the absolute minimum of* A_q *is obtained at the point (or points)* q_0 *satisfying* $\beta(e^{-q_0}) = 1/2$.

Proof. (5.1) follows from the inequality (3.2). (5.2) follows from the fact that $\log q$ is a continuous function and that an integral is a continuous function of its limits. (5.3) follows from the fact that

$$\lim_{q \to \infty} \log q = +\infty ; \lim_{q \to \infty} \int_0^{e^{-q}} \frac{\beta(t)}{t \log 1/t} dt = 0 .$$

We prove (5.4) in the following way: By (2.2) $\beta(1) = \beta(1-0) = 1$ and $\beta(t) \geqq 0$, therefore for sufficiently small $\delta, 0 < \delta < 1$, if $1 - \delta \leqq t \leqq 1$ then $\beta(t) > 2/3$. Therefore for all sufficiently small $q > 0$ we have

$$\begin{aligned} \log q + 2 \int_0^{e^{-q}} \frac{\beta(t)}{t \log 1/t} dt &\geqq \log q + \frac{4}{3} \int_{1-\delta}^{e^{-}} \frac{dt}{t \log 1/t} \\ &= \frac{4}{3} \log \frac{1}{1-\delta} + \frac{1}{3} \log \frac{1}{q} \\ &\to \infty \quad \text{as} \quad q \downarrow 0. \end{aligned}$$

This proves (5.4). (5.5) follows immediately from (5.1)–(5.4). If $\beta(t)$ is continuous then A_q has a continuous derivative

$$\frac{d}{dq} A_q = q^{-1}\{1 - 2\beta(e^{-q})\}$$

and the absolute minimum of A_q, for $q > 0$, is given by $(d/dq)A_q = 0$ or by $\beta(e^{-q}) = 1/2$. This proves (5.6).

Thus we see that for the transformation of Example (5.2)

$$\min_{q>0} A_q = A_{\log 2} = \gamma + \log \log 2 + 2 \int_{\log 2}^{\infty} v^{-1} e^{-v} dv .$$

This result was obtained by R. P. Agnew in [1].

For the transformation of Example (4.3) and for the B_q of Example (4.4) we see that

$$\min_{q>0} A_q = A_{q_0} = \min_{q>0} B_q$$

where q_0 is given by equation

$$\{\Gamma(\alpha+1)\}^{-1}\int_0^{q_0} t^{\alpha}e^{-t}dt = \frac{1}{2}.$$

6. Conclusion. We saw in § 5 that the function A_q (of q) obtained in Theorem (3.1) has an absolute minimum for some $q_0 > 0$. We shall denote this minimum by $B = B(f(x))$. That is $B = A_{q_0}$.

Denote by z' a limit point of a sequence $\{s_n\}$. We denote by z'' a limit point of a linear transform of $\{s_n\}$. Then we obtain from Theorem (3.1) the following result concerning limit points z' and z''.

THEOREM (6.1). *If the suppositions of Theorem* (3.1) *are satisfied then for any sequence* $\{s_n\}$ *satistying* (1.1) *and its* $[J, f(x)]$ *transform we have*: (i) *To each* z' *corresponds at least one* z'' *such that*

$$|z' - z''| \leqq B \cdot \limsup_{n\to\infty} |na_n| . \tag{6.1}$$

(ii) *To each* z'' *corresponds at least one* z' *such that*

$$|z'' - z'| \leqq B \cdot \limsup_{n\to\infty} |na_n| . \tag{6.2}$$

We do not know if the constant B in (6.1) and in (6.2) is the best possible (the smallest).

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