TORSION-FREE MODULES OVER $K[x, y]$

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1. Introduction. Let $R = K[x, y]$ be the ring of polynomials in two variables x and y over a field K. In this note we shall consider the following question: What conditions must be satisfied by two torsion-free R-modules[1] A and B in order that there exist a third R-module C such that $A \oplus C \approx B \oplus C$? Our principal result is the following theorem.

THEOREM. *The following statements are equivalent*:

(a) *There exists an R-module C (not necessarily torsion-free) such that $A \oplus C \approx B \oplus C$.*

(b) $A \oplus R \approx B \oplus R$.

(c) *For any maximal ideal M in R, $A_M \approx B_M$ as R_M-modules.*

(d) *For any maximal ideal M in R, $\bar{A}_M \approx \bar{B}_M$ as $\bar{R}_M$-modules.*

In (c) and (d) above, R_M is the ring of quotients of R with respect to the maximal ideal M, $\bar{R}_M$ is the completion of the local ring R_M, and A_M, $\bar{A}_M$ are the R_M and $\bar{R}_M$-modules, respectively, constructed from A in the standard way. We shall adhere to this notation throughout the paper.

It is natural to ask whether the conditions of the above theorem imply that $A \approx B$, as is trivially the case for the ring of polynomials in one variable. It is perhaps curious that the answer here depends upon the field K. We show that, if K is algebraicly closed of characteristic zero, then A and B satisfy conditions (a) — (d) above if and only if $A \approx B$. However, we provide an example to show that this is not the case if K is the real number field.

The proofs of the preceding statements are based primarily upon the theorem of Seshadri [6] that projective R-modules are free, together with some results of Auslander-Buchsbaum-Goldman ([1], [2]) on duality of modules over commutative Noetherian domains. These will be explained in the next section.

2. Some remarks on duality. Throughout this section R may be any commutative Noetherian normal domain. If A is an R-module, we define $A^* = \mathrm{Hom}_R(A, R)$; A^* will be called the *dual* of A. If B is a second R-module and $f: A \rightarrow B$ is a homomorphism, we shall denote by f^* the induced homomorphism of B^* into A^*. For the basic properties

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[1] Throughout this note, all modules which we consider will be assumed to be finitely generated.

of this functor we refer the reader to [4], p. 476. We shall denote the natural mapping of A^{**} by i_A. If A is torsion-free, then i_A is a monomorphism. In this case we shall consistently identify A with its image in A^{**}. A will be called *reflexive* in case $A = A^{**}$. It is not hard to show that every dual is reflexive; this follows essentially from the fact that, if A is torsion-free, then A and A^* have the same rank.

The following proposition is essentially due to Auslander-Buchsbaum-Goldman ([1], Proposition 3.4, p. 758.)

PROPOSITION 2.1. Let A, B be torsion-free R-modules with the same rank, and assume $A \subseteqq A^{**} \subseteqq B$, $A \neq B$. Let I be the annihilator of B/A (note that $I \neq 0$, since A and B have the same rank.) Then

(a) If $A^{**} = B$, $\operatorname{rank}(I) > 1$.

(b) If $A^{**} \neq B$, $\operatorname{rank}(I) = 1$.

Proof. Assume $\operatorname{rank}(I) = 1$, in which case there exists a prime ideal P in R of rank one such that $I \subseteqq P$. Then $A_P \subsetneqq B_P$. Since R is normal and $\operatorname{rank}(P) = 1$, R_P is a Dedekind ring. Then A_P, being a torsion-free R_P-module, is projective, and therefore trivially reflexive. It then follows from an easy localization argument that $(A^{**})_P = (A_P)^{**} = A_P \subsetneqq B_P$, and therefore $A^{**} \subsetneqq B$. Hence, if $A^{**} = B$, then $\operatorname{rank}(I) > 1$, completing the proof of (a).

Suppose now that $A^{**} \neq B$, and let J be the annihilator of B/A^{**}. We may then apply Proposition 3.4 of [1] (p. 758) to conclude that $\operatorname{rank}(J) = 1$. Since $0 \subsetneqq I \subseteqq J$, it follows that $\operatorname{rank}(I) = 1$, completing the proof of (b).

COROLLARY. *Let B be a reflexive R-module, and A_1, A_2 be submodules of B with same rank as B. Let I_1 and I_2 be the annihilators of B/A_1 and B/A_2, respectively. If the ranks of both ideals are greater than one, then any isomorphism between A_1 and A_2 can be extended to an automorphism of B.*

Proof. Since B is reflexive, we have that $A_1 \subseteqq A_1^{**} \subseteqq B$, $A_2 \subseteqq A_2^{**} \subseteqq B$. But since $\operatorname{rank}(I_1) > 1$, we obtain from Proposition 2.1 that $A_1^{**} = B$, and similarly $A_2^{**} = B$. Hence, if $\theta_1 \colon A_1 \to A_2$ is an isomorphism, then θ_1^{**} is an endomorphism of B. Let $\theta_2 = \theta_1^{-1}$; then θ_2^{**} is likewise an endomorphism of B. Also, $\theta_2^{**}\theta_1^{**} = (\theta_2\theta_1)^{**}$ induces the identity automorphism on A_1. Since B is torsion-free and B/A_1 is a torsion module, it then follows trivially that $\theta_2^{**}\theta_1^{**}$ is the identity on all of B. So is $\theta_1^{**}\theta_2^{**}$, by similar reasoning. Therefore θ_1^{**} is the desired extension of θ_1 to an automorphism of B.

3. Torsion-free modules over regular rings of dimension two. We shall begin this section with a few preliminary results which will prepare the ground for the proof of the theorem mentioned in the introduction.

A square matrix over a ring R will be called a *transvection* if its diagonal entries are all "ones" and there is at most one nonzero entry off the diagonal.

LEMMA 3.1. *Let $R = R_1 \oplus \cdots \oplus R_r$, where each R_i is a local ring. Then any unimodular matrix over R is a product of transvections.*

Proof. Let $A = (a_{ij})$ be a unimodular n-by-n matrix over R. We first consider the special case $r = 1$; i.e., R is a local ring. Then every row and column of A must contain a unit. From this we see easily that A may be reduced to a diagonal matrix by means of standard row and column operations which are equivalent to multiplication by transvections. That is, $A = TDU$, where T, U are products of transvections and—

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \qquad d_i \in R \qquad\qquad d_1 \cdots d_n = 1 \, .$$

We may then apply a well-known trick and write—

$$D = \begin{pmatrix} d_1 & & & & 0 \\ & d_1^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & \cdot \end{pmatrix} \begin{pmatrix} 1 & & & & & 0 \\ & d_1 d_2 & & & & \\ & & (d_1 d_2)^{-1} & & & \\ & & & 1 & & \\ 0 & & & & & \cdot \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & d_1 \cdots d_{n-1} & \\ 0 & & & & d_n \end{pmatrix} .$$

But it is trivial to verify that each of the factors of the above expression is a product of transvections. Thus A is a product of transvections, and the lemma is true for $r = 1$.

Proceed by induction on r; assume $r > 1$ and the lemma is true for $k > r$. Let $R_0 = R_1 \oplus \cdots \oplus R_{r-1}$; then $R = R_0 \oplus R_r$. Let e_0, e_r be the units of R_0 R_r, respectively; then $e_0 + e_r = 1$. Also $A = A_0 + A_r$, where A_0, A_r are unimodular matrices over R_0, R_r, respectively. We have from the induction assumption that $A_0 = \prod_{j=1}^{m} T_0^{(j)}$ and $A_r = \prod_{j=1}^{m} T_r^{(j)}$, where $T_0^{(j)}$ and $T_r^{(j)}$ are transvections over R_0 and R_r, respectively. But then $e_r I + T_0^{(j)}$ and $e_0 I + T_r^{(j)}$ are transvections over R, and it is easy to see that—

$$e_r I + A_0 = \prod_{j=1}^{m} (e_r I + T_0^{(j)}) \qquad e_0 I + A_r = \prod_{j=1}^{m} (e_0 I + T_r^{(j)}) \, .$$

Since $A = A_0 + A_r = (e_r I + A_0)(e_0 I + A_r)$, it is clear that A is a product of transvections, completing the proof.

LEMMA 3.2. *Let R be a direct sum of a finite number of local rings, and F be a free R-module. Let A, B be submodules of F such that $F/A \approx F/B$. Then there exists an automorphism θ of F such that $\theta(A) = B$.*

Proof. If R is a local ring, the lemma follows directly from standard facts concerning minimal epimorphisms ([4], p. 471.) The general case may be deduced from this special case by an easy direct sum argument.

LEMMA 3.3. *Let R be a commutative Noetherian domain. Let F be a free R-module, and A, B be submodules of F, both having the same rank as F. Assume $F/A \approx F/B$, and every prime ideal of R belonging to A (as a submodule of F) is maximal. Then there exists an automorphism θ of $F \oplus R$ such that $\theta(A \oplus R) = B \oplus R$.*

Proof. Let I be the annihilator of F/A (hence also of F/B). Then $IF \subseteqq A \cap B$, and we have the following exact sequences of modules over the ring R/I.

$$0 \longrightarrow A/IF \longrightarrow F/IF \longrightarrow F/A \longrightarrow 0$$
$$0 \longrightarrow B/IF \longrightarrow F/IF \longrightarrow F/B \longrightarrow 0 .$$

Now, it follows from our hypotheses that $\mathrm{Rad}(I) = M_1 \cap \cdots \cap M_r$, where M_i is a maximal ideal in R. Hence we obtain from a direct application of the Chinese Remainder Theorem that R/I is a direct sum of local rings. Therefore, by Lemma 3.2, there exists an automorphism ψ of F/IF such that $\psi(A/IF) = B/IF$. It is easy to see that ψ may be extended to a unimodular automorphism ψ_1 of $(F/IF) \oplus (R/I)$ such that $\psi_1\{(A/IF) \oplus (R/I)\} = (B/IF) \oplus (R/I)$. By Lemma 3.1, ψ_1 is a product of transvections, and thus it is clear that there exists an R-automorphism θ of $F \oplus R$ such that $f\theta = \psi_1 f$, where $f\colon F \oplus R \to (F/IF) \oplus (R/I)$ is the canonical mapping. It then follows immediately that $\theta(A \oplus R) = B \oplus R$, completing the proof of the lemma.

We shall also have use for the following proposition, which was communicated to me by R. Swan.

PROPOSITION 3.4. If R is a complete local ring, then the Krull-Schmidt-Remak Theorem [3] is satisfied by finitely-generated R-modules.

Proof. According to Azumaya's generalization of Krull-Schmidt-Remak Theorem [3], we need only show that, if A is an indecomposable R-module, then the nonunits in $S = \mathrm{Hom}_R(A, A)$ form an ideal. S is a finitely generated R-algebra, and S/MS is an R/M-algebra of finite degree, where M is the maximal ideal in R. If $\bar{e}$ is an idempotent in S/MS, then since R is complete it follows from a standard argument that there exists an idempotent e in S mapping on $\bar{e}$. But $e = 1$ because A is indecomposable, and therefore $\bar{e}$ is the identity of S/MS. We have thus shown that S/MS has a single maximal ideal. Since MS is contained in every maximal ideal of S, we have shown that S itself has a single maximal ideal, and the proposition follows immediately.

Swan, in unpublished work, has shown that Proposition 3.4 does not necessarily hold for incomplete local rings. However, all local rings satisfy a weaker form of the proposition, a fact which is implicit in [3]. For completeness we shall exhibit a proof here.

PROPOSITION 3.5. Let R be a local ring with maximal ideal M, and A and B be R-modules. If there exists a (finitely-generated) free R-module F such that $A \oplus F \approx B \oplus F$, then $A \approx B$.

If A is an R-module, define $d(A)$ to be the dimension of A/MA over the residue class field R/M. Let $\mathscr{C}$ be the class of all R-modules A with the property that there exist R-modules B and F, with F free, such that $A \oplus F \approx B \oplus F$ but $A \neq B$. The proposition simply asserts that $\mathscr{C}$ is empty. Assume the proposition is false; then we may select A from the class $\mathscr{C}$ such that $d(A)$ is minimal. Having fixed A and its companion B, we may then choose F to have minimal rank $n > 0$. Set $C = A \oplus F$; then we may assume that $A, B \subseteqq C$ and there exist free submodules F_1, F_2 of C such that $F_1 \approx F \approx F_2$ and $A \oplus F_1 = C = B \oplus F_2$. Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be bases of F_1 and F_2, respectively. Then there exist homomorphisms f and g of C into R such that $f(A) = g(B) = 0$, $f(x_n) = g(y_n) = 1$, and $f(x_i) = g(y_i) = 0$ for $i < n$. Suppose that $f(F_2) \subseteqq M$, $g(F_1) \subseteqq M$; then, since R is a local ring, it is clear that $f(B) = g(A) = R$. That is, there exist $x \in A$, $y \in B$, such that $f(y) = g(x) = 1$, in which case there exist submodules $A' \subseteqq A$, $B' \subseteqq B$ such that $A = A' \oplus Rx$, $B = B' \oplus Ry$. From this it follows that $A' \oplus R \oplus F \approx A \oplus F \approx B \oplus F \approx B' \oplus R \oplus F$. But $d(A') = d(A) - 1$, and hence $A' \approx B'$, since A was chosen from the class $\mathscr{C}$ so that $d(A)$ is minimal. But then $A \approx A' \oplus R \approx B' \oplus R \approx B$, a contradiction. Therefore we may assume that either $f(F_2) = R$ or $g(F_1) = R$; let us say that $f(F_2) = R$. Then $f(y_i)$ is a unit for some $i \leqq n$, say $i = 1$. Define a homomorphism $j: R \to C$ by $j(a) = a(f(y_1))^{-1}y_1$, where $a \in R$; then it is clear that fj is the identity map on R. We leave to the reader the trivial verification of the resulting fact that $A \oplus F' \approx \ker(f) \approx \operatorname{coker}(j) \approx B \oplus F'$, where F' is a free R-module of rank $n - 1$. But this contradicts the fact that F was chosen to be the free module of minimal rank with the property that $A \oplus F \approx B \oplus F$. The proof of the proposition is hence complete.

We are now ready to prove a slight generalization of the theorem stated in the introduction.

THEOREM 3.6. *Let R be a commutative Noetherian domain. Assume that the global dimension of R is less than or equal to two, and every projective R-module is free. Let A, B be torsion-free R-modules. Then the following statements are equivalent—*

(a) *There exists an R-module C such that $A \oplus C \approx B \oplus C$.*

(b) $A \oplus R \approx B \oplus R$.

(c) $A_M \approx B_M$ *as R_M-modules for every maximal ideal M in R.*

(d) $\bar{A}_M \approx \bar{B}_M$ *as $\bar{R}_M$-modules for every maximal ideal M in R.*

Proof. (a)$\Rightarrow$(d): If $A \oplus C \approx B \oplus C$, then certainly $\bar{A}_M \oplus \bar{C}_M \approx \bar{B}_M \oplus \bar{C}_M$ for any maximal ideal M in R. It then follows from Proposition 3.4 that $\bar{A}_M \approx \bar{B}_M$.

(b)$\Rightarrow$(a): Obvious.

(c)$\Rightarrow$(d): Obvious.

(b)$\Rightarrow$(c): If $A \oplus R \approx B \oplus R$, then $A_M \oplus R_M \approx B_M \oplus R_M$ for any maximal ideal M in R. We may then apply Proposition 3.5 to conclude that $A_M \approx B_M$.

(d)$\Rightarrow$(b); If (d) holds, we have immediately that A and B have the same rank. If A is projective, it follows from a standard result of homological algebra that B is likewise projective, in which case both are free by hypothesis and (b) follows trivially. Thus we may assume that neither A nor B is projective. Since gl.dim.$(R) \leqq 2$, we obtain from the Corollary to Proposition 4.7 of [2] (p. 17) that A^{**} and B^{**} are projective (the hypothesis given there that R be local is easily seen to be unnecessary. This fact also follows, perhaps more simply, from (4.4) of [4], p. 477.) Our hypotheses then imply that A^{**} and B^{**} are free; and, of course, they have the same rank. We may then identify A^{**} and B^{**}, and write $A^{**} = B^{**} = F$, a free R-module. $A \subseteqq F$, $B \subseteqq F$, and if I and J are the annihilators of F/A and F/B, respectively, then it follows from Proposition 2.1 that both ideals have rank greater than one (we should remark at this point that R is normal, since it has finite global dimension; hence the hypotheses of Proposition 2.1 are satisfied.)

Let M be a maximal ideal in R; then by hypothesis $\bar{A}_M \approx \bar{B}_M$. IR_M and JR_M are the annihilators of $\bar{F}_M/\bar{A}_M$ and $\bar{F}_M/\bar{B}_M$, respectively, and both of these ideals in $\bar{R}_M$ have rank greater than one. Furthermore, since R has finite global dimension, $\bar{R}_M$ is a regular local ring, and so we may apply the Corollary to Proposition 2.1 to conclude that there exists an $\bar{R}_M$-automorphism φ of $\bar{F}_M$ such that $\varphi(\bar{A}_M) = \bar{B}_M$. In particular, $(\overline{F/A})_M \approx \bar{F}_M/\bar{A}_M \approx \bar{F}_M/\bar{B}_M \approx (\overline{F/B})_M$. Now, since rank$(I) > 1$ and Krull dim.(R) = gl.dim.$(R) \leqq 2$, we obtain easily from the Chinese Remainder Theorem that R/I is a direct sum of local rings, each with nilpotent maximal ideal. Then, since $(\overline{F/A})_M$ and $(\overline{F/B})_M$ may be viewed as modules over $\bar{R}_M/I\bar{R}_M \approx R_M/IR_M$, it follows from standard properties of completions of local rings that $(F/A)_M \approx (F/B)_M$. This is true for every maximal ideal M in R, and hence $F/A \approx F/B$ as R-modules, since both may be viewed as modules over R/I, a direct sum of local rings. Since every prime ideal in R belonging to A or B (as a submodule of F) is maximal, we may apply Lemma 3.3 to conclude that there exists an automorphism θ of $F \oplus R$ such that $\theta(A \oplus R) = B \oplus R$. In particular, $A \oplus R \approx B \oplus R$, completing the proof of the theorem.

COROLLARY. *If $R = K[x, y]$, K a field, then R satisfies the conditions of Theorem* 3.6.

Proof. The well-known fact that gl.dim.$(R) = 2$ ([5], p. 180), together with Seshadri's result [6] that projective R-modules are free, imply that R satisfies the hypotheses, and hence the conclusions, of Theorem 3.6.

As mentioned in the introduction, we are able to improve Theorem 3.6 for $R = K[x, y]$ if certain assumptions are made concerning the field K.

THEOREM 3.7. *Let $R = K[x, y]$, where K is an algebraicly closed field of characteristic p. Let A, B be torsion-free R-modules of the same rank n. If p does not divide n, then A and B satisfy the conditions of Theorem* 3.6 *if and only if $A \approx B$.*

Proof. As in Theorem 3.6, we may assume that neither A nor B is projective, but both are contained in a free R-module F in such a way that $F/A \approx F/B$. Furthermore, if I is the annihilator of F/A (hence also of F/B) then $R/I = R_1 \oplus \cdots \oplus R_r$, where R_i is a local ring with nilpotent maximal ideal M_i. Let e_i be the unit of R_i and $\bar{e}_i$ be the unit of R_i/M_i. Since K is algebraicly closed, $R_i/M_i = K\bar{e}_i$.

Now, F/IF is a free R/I-module, and so we may apply Lemma 3.2 to obtain an automorphism θ of F/IF such that $\theta(A/IF) = B/IF$. Write $\theta_i = e_i\theta$; then $\theta = \theta_1 + \cdots + \theta_r$. If $d_i = \det(\theta_i)$, then $d_1 + \cdots + d_r = d = \det(\theta)$. d is a unit in R/I, and d_i is a unit in R_i. Since $R_i/M_i = K\bar{e}_i$, we may write $d_i = a_i(e_i + u_i)$, where $a_i \in K$ and $u_i \in M_i$. Since K is algebraicly closed, there exist $b_i \in K$ such that $b_i^n = a_i^{-1}$. Since M_i is nilpotent, we see immediately that the multiplicative group of units of R_i which map on $\bar{e}_i$ has exponent a power of p, and therefore, since p does not divide n, there exist $c_i \in R_i$ such that $c_i^n = (e_i + u_i)^{-1}$. Set

$\theta' = b_1c_1\theta_1 + \cdots + b_rc_r\theta_r = (b_1c_1 + \cdots + b_rc_r)\theta$; then θ' is a unimodular automorphism of F/IF and $\theta'(A/IF) = B/IF$. By Lemma 3.1, θ' is a product of transvections, and thus there exists an R-automorphism φ of F such that $\theta' f = f\varphi$, where $f\colon F \to F/IF$ is the canonical mapping. Since $IF \subseteqq A \cap B$, it follows easily that $\varphi(A) = B$. Therefore $A \approx B$, completing the proof of the theorem.[2]

4. Examples. In this section we shall show that $R = K[x, y]$ does not satisfy Theorem 3.7 if K is the field of real numbers.

LEMMA 4.1. *Let $S = K[x, y]/((x^2 - 1)^3, (x^2 - 1)^2y^2, y^3)$, where K is the real number field. Set $F = S \oplus S$, and define submodules A and B of F to be generated by the rows of the following matrices—*

$$A\colon \begin{pmatrix} (x^2-1)^2 & 0 \\ 0 & y^2 \\ y & x^2-1 \end{pmatrix} \qquad B\colon \begin{pmatrix} x(x^2-1)^2 & 0 \\ 0 & y^2 \\ xy & x^2-1 \end{pmatrix}$$

Then there exists no automorphism θ of F such that $\theta(A) = B$ and $\det(\theta) \in K$.

Proof. Set $P_1 = (x - 1, y) \subseteqq S$, $P_2 = (x + 1, y) \subseteqq S$, and $Q = P_1 \cap P_2 = (x^2 - 1, y)$; then Q is easily seen to be the radical of S, and $S/Q \approx S/P_1 \oplus S/P_2 \approx K \oplus K$. $(1 + x)/2$ and $(1 - x)/2$ are orthogonal idempotents modulo Q, and therefore it is clear that any u in S can be expressed in the form $u = \lambda(x + 1) + \mu(x - 1) + u'$, where $u' \in Q$ and $\lambda, \mu \in K$.

We assert first that $\{(x + 1)(x^2 - 1)y^2, 0\}$, $\{(x - 1)(x^2 - 1)y^2, 0\}$, $\{0, (x + 1)(x^2 - 1)^2y\}$, and $\{0, (x - 1)(x^2 - 1)^2y\}$ are not in A. For suppose $\{(x + 1)(x^2 - 1)y^2, 0\}$ is in A; then

$$\begin{aligned} \{(x + 1)(x^2 - 1)y^2, 0\} &= p\{(x^2 - 1)^2, 0\} + q\{0, y^2\} + r\{y, x^2 - 1\} \\ &= \{p(x^2 - 1)^2 + ry, qy^2 + r(x^2 - 1)\} \end{aligned}$$

for some p, q, r in S. Then $(x + 1)(x^2 - 1)y^2 = p(x^2 - 1)^2 + ry$, from which it follows that $r = -(x + 1)(x^2 - 1)y + r'(x^2 - 1)^2 + r''$, where $r' \in S$ and $r'' \in Q^3$. But then

$$\begin{aligned} 0 = qy^2 + r(x^2 - 1) &= qy^2 - (x + 1)(x^2 - 1)^2y + r'(x^2 - 1)^3 + r''(x^2 - 1) \\ &= qy^2 - (x + 1)(x^2 - 1)^2y\ , \end{aligned}$$

since $(x^2 - 1)^3 = Q^4 = 0$. But this equation is easily seen to be impossible, and so we have that $\{(x + 1)(x^2 - 1)y^2, 0\}$ is not in A. The other

[2] The proof of Theorem 3.7 has been phrased for $p>0$. However, the theorem is also true if $p=0$, since then the binomial theorem may be used to obtain $c_i \in R_i$ such that $c_i^n = (e_i + u_i)^{-1}$.

assertions can be proved in similar fashion.

Suppose now that there exists an automorphism θ of F such that $\theta(A) = B$ and $\det(\theta) = t \in K$. Define a mapping $\tau: F \to F$ by $\tau(\{u, v\}) = \{xu, v\}$. τ is an endomorphism of F with determinant x. But $x = (1 + x)/2 - (1 - x)/2$ is a unit modulo Q, and hence is a unit in S, since Q is the radical of S. Therefore τ is an automorphism of F. Clearly $\tau(A) = B$. Set $\sigma = \theta^{-1}\tau$; then, replacing t by t^{-1}, we get that σ is an automorphism of F with determinant tx, and $\sigma(A) = A$. Relative to the given basis of F, σ may be represented by a matrix—

$$\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \qquad a, b, c, d \in S \qquad ad - bc = tx$$

From the equation—

$$\begin{pmatrix} (x^2 - 1)^2 & 0 \\ 0 & y \\ y & x^2 - 1 \end{pmatrix} \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} a(x^2 - 1)^2 & b(x^2 - 1)^2 \\ cy^2 & dy^2 \\ ay + c(x^2 - 1) & by + d(x^2 - 1) \end{pmatrix}$$

it follows that $\{0, b(x^2 - 1)^2\}$ and $\{cy^2, 0\}$ are in A. Write $b = \lambda(x + 1) + \mu(x - 1) + b'$, where $\lambda, \mu \in K$ and $b' \in Q$; then, since $Q^4 = 0$ and $((x + 1)/2)(x + 1) \equiv x + 1 \pmod Q$, we have that $\{0, \lambda(x + 1)(x^2 - 1)^2 y\} = \{0, ((x + 1)/2)b(x^2 - 1)^2 y\} \in A$. If $\lambda \neq 0$, then $\{0, (x + 1)(x^2 - 1)^2 y\} \in A$, contradicting our previous remarks. Hence $\lambda = 0$. A similar argument shows that $\mu = 0$. Therefore $b \in Q$, in which case $b = b_1(x^2 - 1) + b_2 y$, where $b_1, b_2 \in S$. It follows from similar reasoning that $c = c_1 y + c_2(x^2 - 1)$, where $c_1, c_2 \in S$.

We then see that

$$\begin{aligned} &\{ay + c(x^2 - 1),\ by + d(x^2 - 1)\} \\ &\quad = \{ay + c_1(x^2 - 1)y + c_2(x^2 - 1)^2,\ b_1(x^2 - 1)y + b_2 y^2 + d(x^2 - 1)\} \end{aligned}$$

is in A, and then $\{y[a + c_1(x^2 - 1)],\ (x^2 - 1)[b_1 y + d]\}$ is in A, since $\{(x^2 - 1)^2, 0\}$ and $\{0, y^2\}$ are in A. Therefore

$$\begin{aligned} w &= \{0, (x^2 - 1)[b_1 y - c_1(x^2 - 1) + (d - a)]\} \\ &= \{y[a + c_1(x^2 - 1)],\ (x^2 - 1)(b_1 y + d)\} - [a + c_1(x^2 - 1)]\{y, x^2 - 1\} \end{aligned}$$

is in A. Write $d - a = \lambda(x + 1) + \mu(x - 1) + u$, where $\lambda, \mu \in K$ and $u \in Q$. Then, using once again the facts that $(x + 1)/2$ and $(x - 1)/2$ are orthogonal idempotents modulo Q and $Q^4 = 0$, we obtain that $\{0, \lambda(x + 1)(x^2 - 1)^2 y\} = ((1 + x)/2)(x^2 - 1)w \in A$, and hence $\lambda = 0$, since $\{0, (x + 1)(x^2 - 1)^2 y\}$ is not in A. $\mu = 0$ for similar reasons, and therefore $d - a \in Q$; i.e., $a \equiv d \pmod Q$. But then $tx = ad - bc \equiv ad \equiv a^2 \pmod Q$, since $b, c \in Q$. Recall now that $S/Q = K_1 \oplus K_2$, where $K_1 \approx K \approx K_2$. Let $\varepsilon_1, \varepsilon_2$ be the units of K_1, K_2, respectively; then, under the isomorphism just mentioned, $(1+x)/2$ maps onto ε_1 and $(1-x)/2$ maps onto ε_2, in which case $x=(1+x)/2-(1-x)/2$ maps onto $\varepsilon_1-\varepsilon_2$. We have thus shown that there exists $\alpha \in K_1 \oplus K_2$ such that $\alpha^2 = t\varepsilon_1 - t\varepsilon_2$. This can be true only if both t and $-t$ have square roots in K. But this is impossible unless $t=0$, and so we have reached a contradiction. Therefore θ cannot exist, and the proof of the lemma is complete.

PROPOSITION 4.2. Let $R=K[x, y]$, where K is the field of real numbers, and set $I=((x^2-1)^3, (x^2-1)^2y^2, y^3)$, an ideal in R. Let $F=R\oplus R$, and define submodules A', B' of F to be generated by the rows of the following matrices—

$$A'\colon\begin{pmatrix} (x^2-1)^2 & 0 \\ 0 & y^2 \\ y & x^2-1 \end{pmatrix} \qquad B\colon\begin{pmatrix} x(x^2-1)^2 & 0 \\ 0 & y^2 \\ xy & x^2-1 \end{pmatrix}$$

and let $A=A'+IF$, $B=B'+IF$. Then $A\oplus R \approx B\oplus R$, but $A \not\approx B$.

Proof. Set $S=R/I$; then $F/IF \approx S\oplus S$, a free S-module. Define a mapping $\varphi\colon F/IF \to F/IF$ by $\varphi(\{u, v\}) = \{xu, v\}$. φ is an endomorphism of F/IF, and $\det(\varphi)=x$, which is a unit of S; hence φ is an automorphism. Furthermore, $\varphi(A/IF)=B/IF$, from which it follows that $F/A\approx F/B$. Therefore, $A\oplus F\approx B\oplus F$, by the the theorem of Schanuel [7]. We may then apply Theorem 3.6 to conclude that $A\oplus R\approx B\oplus R$.

Suppose now that $A\approx B$. It is easy to see that $\operatorname{rank}(I)=2$; hence, since $IF \subsetneqq A\cap B$, we have from the corollary to Proposition 2.1 that the isomorphism between A and B can be extended to an automorphism θ of F. Then $\det(\theta)=t\in K$, since K contains every unit of R. Reducing modulo I, we obtain an automorphism θ' of F/IF such that $\theta'(A/IF)=B/IF$ and $\det(\theta')=t$. But this contradicts Lemma 4.1 as applied to S, F/IF, A/IF, and B/IF. Hence $A\not\approx B$, completing the proof of the proposition.

In closing, we remark that it is not difficult to see that Theorems 3.6 and 3.7 do not hold for a ring of polynomials in more than two variables.

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