ASYMPTOTIC ESTIMATES FOR LIMIT POINT PROBLEMS

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Introduction. The variation of characteristic values and functions of the differential operator L defined by

$$Lx = \frac{1}{k(s)}\left\{-\frac{d}{ds}\left[p(s)\frac{dx}{ds}\right] + q(s)x\right\}$$

will be studied when the domain of L varies because of a change of boundary conditions. The *basic* interval is an open interval $\omega_- < s < \omega_+$ on which k is positive and piecewise continuous, p is positive and differentiable, and q is real-valued and piecewise continuous. For a closed subinterval $[a, b]$ of the basic interval, our purpose is to obtain estimates for the characteristic values μ_{ab} and characteristic functions y_{ab} of regular Sturm-Liouville problems on $[a, b]$ when a, b are near ω_-, ω_+. Such results have been obtained by the author [6] in the case that both ω_- and ω_+ are limit circle singularities in H. Weyl's classification [2, p. 225]. Here the analogous results will be derived in the limit point case and the mixed case (one singularity of each type). To avoid repetition of the preliminary material in [6], we shall usually adhere to the notation and numbering system of [6] without further comment.

6. Basic problems in the limit point and mixed cases. As in § 2, the limits of μ_{ab} as $a \to \omega_-, b \to \omega_+$ are supposed to exist, and accordingly we shall assume that characteristic values λ of suitable singular Sturm-Liouville problems for L on (ω_-, ω_+) exist. These singular problems are described as follows when both ω_-, ω_+ are limit point singularities [4].

Let L_0 be the differential operator $L - l_0$, $Im\, l_0 \neq 0$. According to a theorem of Weyl [4, p. 45] there exist linearly independent solutions φ_-, φ_+ of $L_0\varphi = 0$ such that

$$\varphi_+ \in \mathfrak{F}_{\omega\omega_+}\,, \qquad \varphi_- \in \mathfrak{F}_{\omega_-\omega}\,, \qquad [\varphi_+\bar{\varphi}_-](s) = 1 \tag{6.1}$$

for any ω satisfying $\omega_- < \omega < \omega_+$. These solutions are uniquely determined from the normalization condition $[\varphi_+\varphi_+](s_0) = i$ at some point s_0, to remain fixed in the sequel. (Compare (6.1) with the choice (2.1) of φ_-, φ_+ in the limit circle case.) Let $\mathfrak{D}^0$ be the set of all x in the basic Hilbert space $\mathfrak{H}$ (described in § 1) which have the following properties: (a) x is differentiable on (ω_-, ω_+) and x' is absolutely

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continuous on every closed subinterval of this interval; and (b) $Lx \in \mathfrak{H}$. The basic characteristic value problem in the limit point case is then

$$Lx = \lambda x\ , \qquad x \in \mathfrak{D}^0\ . \tag{6.2}$$

In this case, x is not restricted by any boundary conditions at ω and ω_+.

Our main assumption is that there exists at least one characteristic value λ of this problem. It will be supposed that a corresponding characteristic function x has been selected with $\|x\| = 1$.

In the limit circle case, no special assumptions on L at ω_- and ω_+ had to be imposed, but the generality of the boundary operators U_a and U_b [See (1.5), (2.4)] had to be sacrificed in order to ensure that $\mu_{ab} \to \lambda$ as $[a, b] \to (\omega_-, \omega_+)$. In the limit point case herein under consideration, the situation is quite different. Some additional restrictions on L as $s \to \omega_\pm$ are clearly needed to get a point spectrum at all, but then very general boundary operators U_a, U_b will permit convergence of μ_{ab} to λ. The following notation will be used:[1]

$$\sigma_a = \varphi_-(a)/\varphi_+(a)\ ; \qquad \sigma_b = \varphi_+(b)/\varphi_-(b)\ ; \tag{6.3}$$

$$\xi_a = [x(a)/\varphi_+(a)]\, \|\varphi_+\|_a\ ; \qquad \xi_b = [x(b)/\varphi_-(b)]\, \|\varphi_-\|^b\ ; \tag{6.4}$$

$$\xi_a^* = \sigma_a \|\varphi_+\|_a\ ; \qquad \xi_b^* = \sigma_b \|\varphi_-\|^b\ ; \tag{6.5}$$

$$\eta_a = U_a\varphi_-/U_a\varphi_+\ ; \qquad \eta_b = U_b\varphi_+/U_b\varphi_-\ ; \tag{6.6}$$

$$\theta_a = (U_a x/U_a\varphi_+)\, \|\varphi_+\|_a\ ; \qquad \theta_b = (U_b x/U_b\varphi_-)\, \|\varphi_-\|^b\ ; \tag{6.7}$$

$$\theta_a^* = \eta_a \|\varphi_+\|_a\ ; \qquad \theta_b^* = \eta_b \|\varphi_-\|^b\ ; \tag{6.8}$$

$$\omega_- < a \leqq a_0\ , \qquad b_0 \leqq b < \omega_+\ .$$

The assumptions below turn out to be sufficient for $\mu_{ab} - \lambda$ and $\|y_{ab} - x\|_a^b$ to be $o(1)$ as $[a, b] \to (\omega_-, \omega_+)$.

ASSUMPTIONS. (*ω_- and ω_+ limit point singularities*)

(i) *The singularities ω_- and ω_+ are not accumulation points of the zeros of $\varphi_\pm$ and*

$$\xi_a = o(1) \quad \textit{and} \quad \xi_a^* = o(1) \quad \textit{as } a \to \omega_-\ ; \tag{6.9}$$

$$\xi_b = o(1) \quad \textit{and} \quad \xi_b^* = o(1) \quad \textit{as } b \to \omega_+\ . \tag{6.10}$$

(ii) *The boundary operators U_a, U_b are restricted only by the boundedness of the quantities*

[1] The abbreviations $\|\varphi\|_a$, $\|\varphi\|^b$ are used for $\|\varphi\|_a^{\omega_+}$, $\|\varphi\|_{\omega_-}^b$, following the convention of § 1.

$$(6.11)\qquad \begin{matrix} \varphi_+(a)U_a\varphi_-/\varphi_-(a)U_a\varphi_+ \ ; & \varphi_+(a)U_ax/x(a)U_a\varphi_+ \ ; \\ \varphi_-(b)U_b\varphi_+/\varphi_+(b)U_b\varphi_- \ ; & \varphi_-(b)U_bx/x(b)U_b\varphi_- \end{matrix}$$

in some neighborhoods $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$ *of* ω_-, ω_+ *respectively.*

According to (6.3)–(6.8), these assumptions imply

$$(6.12)\qquad \sigma_s = o(1),\ \eta_s = o(1),\ \theta_s = o(1),\ \theta_s^* = o(1) \quad \text{as } s \to \omega_\pm \ .$$

The weaker assumptions $\theta_s = o(1)$, $\theta_s^* = o(1)$ in (6.12) are actually sufficient for Theorem 4, while the stronger assumptions (6.9)–(6.11) are needed for the uniform estimate of Theorem 5.

It follows from (6.3), (6.6), (6.11), and (6.12) that there exist constants a_0, b_0, and C such that

$$(6.13)\qquad |\sigma_a| \leqq 1,\ |\sigma_b| \leqq 1,\ |\eta_a| \leqq C|\sigma_a|,\ |\eta_b| \leqq C|\sigma_b|$$

provided $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$, and

$$(6.14)\qquad \begin{cases} |\sigma_a| \leqq |\sigma_s| & \text{if } \omega_- < a \leqq s \leqq a_0 \ ; \\ |\sigma_b| \leqq |\sigma_s| & \text{if } b_0 \leqq s \leqq b < \omega_+ \ . \end{cases}$$

Condition (ii) above (6.11) is only a slight restriction on the boundary operators U_a, U_b. Compare (2.4) and (5.2) for limit circle problems of class 1 and 2 respectively. Sufficient conditions for the validity of (ii) when ω_- is a regular singularity or an irregular singularity of finite rank are stated in [**5**, p. 840, p. 844]. In particular when $\omega_- = 0$ is a regular singularity of L_0 with real, distinct exponents, then a sufficient condition for (ii) is that $\lim\,[-a\alpha_0(a)/\alpha_1(a)]$ $(a \to 0)$ exist (finite or ∞) and be different from the smaller exponent.

We shall now describe a basic problem of the mixed type. It is enough to consider the case that ω_- is a limit circle singularity and ω_+ is a limit point singularity. Then there exist solutions $\varphi_\pm$ of $L_0\varphi = 0$ which satisfy

$$(6.15)\qquad \varphi_+ \in \mathfrak{H},\ \varphi_- \in \mathfrak{F}_{\omega_-\omega},\ [\varphi_-\varphi_-](-) = 0,\ [\varphi_+\bar{\varphi}_-](s) = 1 \ ,$$

where $\omega_- < \omega < \omega_+$, and these solutions will be determined once and for all by the fixed (but arbitrary) normalization $[\varphi_+\varphi_+](s_0) = i$ $(\omega_- < s_0 < \omega_+)$. Thus φ_+ is described by (6.1) and φ_- is described by (2.1) in the mixed case.

Let $\mathfrak{D}^0$ be the basic domain described above (6.2) and let $\mathfrak{D}^1$ be the set of all $x \in \mathfrak{D}^0$ which satisfy the end condition $[x\varphi_-](-) = 0$. The basic characteristic value problem in the mixed case is then

$$(6.16)\qquad Lx = \lambda x \ , \qquad x \in \mathfrak{D}^1 \ .$$

In the mixed case, assumptions (6.10) and the second of (6.11) are in effect at ω_+ together with the first assumption (2.4) at ω_-.

Asymptotic estimates for the difference $\mu_{ab} - \lambda$ between characteristic values of (2.5) and (6.16) when a, b are near ω_-, ω_+ will be obtained in § 9. The limit circle case has already been treated in §§ 3, 4 and the limit point case, when (6.2) replaces (6.16), will be treated in § 7. Also uniform estimates for the difference $y_{ab}(s) - x(s)$ on $a \leqq s \leqq b$ will be obtained under slightly stronger assumptions in §§ 8 and 10. From these results, asymptotic variational formulae for characteristic values will be derived in § 11.

7. Asymptotic estimates in the limit point case at both endpoints. When both ω_- and ω_+ are limit point singularities, the basic problem is (6.2) and (2.5) is regarded as a perturbation of (6.2) arising from adjoining the boundary conditions $U_a y = U_b y = 0$ at $s = a$ and $s = b$. The assumptions (6.9)–(6.11) are used in this section.

Let $G_{ab}(s, t)$ denote the Green's function for the differential operator kL_0 associated with the boundary conditions $U_a y = U_b y = 0$, and let G_{ab} denote the linear integral operator on $\mathfrak{F}_{ab}$ defined by the equation

$$G_{ab}v(s) = \int_a^b G_{ab}(s, t)v(t)k(t)dt,\ v \in \mathfrak{F}_{ab}\ . \tag{7.1}$$

It is well-known [4, p. 20] that for any piecewise continuous function v on $a \leqq s \leqq b$, the function $w = G_{ab}v$ is the unique solution in $\mathfrak{D}_{ab}$ [see (2.5)] of the differential equation $L_0 w = v$.

Let λ be a characteristic value of the basic problem (6.2) and let x be a corresponding normalized characteristic function satisfying (6.9)–(6.11). Define a function f on $[a, b]$ by the equation[2]

$$f = x - \gamma G_{ab}x\ , \quad \text{where } \gamma = \lambda - l_0\ . \tag{7.2}$$

Then f is the unique solution of the boundary value problem $L_0 f = 0$, $U_a f = U_a x$, $U_b f = U_b x$, which has the following representation in terms of the functions φ_-, φ_+ described by (6.1):

$$\begin{aligned} f(s) &= \left(\frac{U_a x}{U_a\varphi_+}\right)\left(\frac{\eta_b\varphi_-(s) - \varphi_+(s)}{\eta_a\eta_b - 1}\right) \\ &\quad + \left(\frac{U_b x}{U_b\varphi_-}\right)\left(\frac{\eta_a\varphi_+(s) - \varphi_-(s)}{\eta_a\eta_b - 1}\right)\ . \end{aligned} \tag{7.3}$$

It follows from (6.7), (6.8) that

$$\|f\|_a^b \leqq |1 - \eta_a\eta_b|^{-1}(|U_a x/U_a\varphi_+|\,|\theta_b^*| + |\theta_a| + |U_b x/U_b\varphi_-|\,|\theta_a^*| + |\theta_b|)$$

[2] The function on $[a, b]$ which coincides with x on this interval will also be denoted by x.

According to (6.12), $\eta_a = o(1)$, $\theta_a = o(1)$, $\theta_a^* = o(1)$ as $a \to \omega_-$ and $\eta_b = o(1)$, $\theta_b = o(1)$, $\theta_b^* = o(1)$ as $b \to \omega_+$. Hence there exists a rectangle R_0 and a constant[3] C on R_0 such that $|\eta_a \eta_b| \leqq \frac{1}{2}$ for $[a, b] \in R_0$, and

$$\|f\|_a^b \leqq C(|\theta_a| + |\theta_b|) \quad \text{for } [a, b] \in R_0 . \tag{7.4}$$

It follows from (7.2) and (7.4) that for any characteristic function x associated with the characteristic value λ,

$$\|x - \gamma G_{ab} x\|_a^b \leqq \mathrm{C}(|\theta_a| + |\theta_b|)\,\|x\| . \tag{7.5}$$

Let $P(\delta)$ $(\delta > 0)$ be the projection from $\mathfrak{F}_{ab}$ onto the subspace $\mathfrak{F}_{ab}(\delta)$ spanned by all characteristic functions y^i of (2.5) whose corresponding μ^i lie in the interval $|\mu^i - \lambda| \leqq \delta$. Then according to the fundamental lemma of § 2,

$$\|x - P(\delta)x\|_a^b \leqq (1 + |\gamma|/\delta)\,\|x - \gamma G_{ab} x\|_a^b .$$

The proof appears in [1]. With the aid of (7.5), we see that there exists a constant C on R_0 such that

$$\|x - P(\delta)x\|_a^b \leqq (C/2\delta)(|\theta_a| + |\theta_b|)\,\|x\|_a^b$$

provided $[a, b] \in R_0$. With the choice $\delta = C(|\theta_a| + |\theta_b|)$ we conclude that $P(C|\theta_a| + C|\theta_b|)x = 0$ implies that $x = 0$ on $[a, b]$. Hence there exists at least one characteristic value $\mu = \mu_{ab}$ of (2.5) such that $|\mu_{ab} - \lambda| \leqq C(|\theta_a| + |\theta_b|)$ if $[a, b] \in R_0$. The proof that there is exactly one follows that in the limit circle case and will be omitted. [6, § 3] The following analogue of Theorem 3 is therefore valid:

THEOREM 4. *If both singularities ω_- and ω_+ of the differential operator L are of the limit point type, under the assumptions* (6.9)–(6.11), (*or even under the weaker assumptions $\theta_s = o(1)$, $\theta_s^* = o(1)$ as $s \to \omega_\pm$*) *then for every basic characteristic value λ of* (6.2) *there exists a rectangle R_0 and a constant C on R_0 such that a unique μ_{ab} satisfies $|\mu_{ab} - \lambda| \leqq C(|\theta_a| + |\theta_b|)$ whenever $[a, b] \in R_0$. There are normalized characteristic functions x, y_{ab} associated with λ, μ_{ab} respectively such that $\|y_{ab} - x\|_a^b \leqq C(|\theta_a| + |\theta_b|)$.*

8. Uniform estimates in the limit point case. In order to obtain uniform estimates for $y_{ab}(s) - x(s)$ on $a \leqq s \leqq b$, following the method of § 4, we need stronger assumptions than (6.9)–(6.11). It will be supposed in addition that the following are bounded on $\omega_- < s < \omega_+$;

$$\varphi_+(s)\,\|\varphi^-\|^s \,; \qquad \varphi_-(s)\,\|\varphi_+\|_s \tag{8.1}$$

[3] C will be used throughout as a generic notation for a constant on R_0.

Let a_0, b_0 be the fixed numbers in (6.11)–(6.14) and let $\hat{\varphi}_\pm(s)$ be defined by

$$\begin{aligned} \hat{\varphi}_\pm(s) &= |\varphi_\pm(s)| \quad \text{if } \omega_- < s < a_0,\, b_0 < s < \omega_+ \\ &= 1 \qquad\qquad \text{if } a_0 \leqq s \leqq b_0 \,. \end{aligned} \tag{8.2}$$

We assert that there exists a constant C, independent of a, b as well as s, such that

$$|\eta_a\varphi_+(s)| \leqq C\hat{\varphi}_-(s) \quad \text{on } a \leqq s \leqq b,\, a \leqq a_0 \,; \tag{8.3}$$

$$|\eta_b\varphi_-(s)| \leqq C\hat{\varphi}_+(s) \quad \text{on } a \leqq s \leqq b,\, b_0 \leqq b \,. \tag{8.4}$$

These inequalities are obvious on the fixed intervals $a_0 \leqq s \leqq b_0$. To complete the proof of (8.4), we deduce from (6.3), (6.13), and (6.14) that

$$|\eta_b\varphi_-(s)| \leqq C|\sigma_b\varphi_-(s)| \leqq C|\sigma_s\varphi_-(s)| = C|\varphi_+(s)|$$

on $b_0 \leqq s \leqq b < \omega_+$. Since $|\sigma_s| = |\varphi_-(s)/\varphi_+(s)| \leqq 1$ on $\omega_- < s \leqq a_0$, by (6.13), it follows that $|\eta_b\varphi_-(s)| \leqq C|\varphi_+(s)|$ on $\omega_- < a \leqq s \leqq a_0$ as well. Thus (8.4) is valid on the whole interval $a \leqq s \leqq b$. The proof of (8.3) is similar and will be omitted.

The Green's function $G_{ab}(s, t)$ for L on $\mathfrak{D}_{ab}$ (associated with the boundary conditions $U_a y = U_b y = 0$) is given by

$$\begin{aligned} G_{ab}(s, t) &= \Omega^{-1}\psi_a(t)\psi_b(s) \quad \text{if } a \leqq t \leqq s \leqq b \,, \\ &= \Omega^{-1}\psi_a(s)\psi_b(t) \quad \text{if } a \leqq s \leqq t \leqq b \,, \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} \psi_a(s) &= \varphi_-(s)U_a\varphi_+ - \varphi_+(s)U_a\varphi_- \,, \\ \psi_b(s) &= \varphi_-(s)U_b\varphi_+ - \varphi_+(s)U_b\varphi_- \,, \\ \Omega &= U_a\varphi_-U_b\varphi_+ - U_a\varphi_+U_b\varphi_- \,. \end{aligned} \tag{8.6}$$

Let G_{ab} denote the Green's operator (7.1). It will first be shown that $\gamma G_{ab}x(s)$ is uniformly close to $y(s)$ on $a \leqq s \leqq b$ when a, b are near ω_-, ω_+. The following lemma will be needed in the proof.

LEMMA 2. *The positive function g_{ab} defined by*

$$g_{ab}^2(s) = \int_a^b |G_{ab}(s, t)|^2 k(t)dt$$

is uniformly bounded on $a \leqq s \leqq b$ provided $a \leqq a_0$, $b_0 \leqq b$.

Proof. According to (6.6), (8.5), and (8.6), $g_{ab}(s)$ has the following explicit representation

$$g_{ab}^2(s) = |1 - \eta_a\eta_b|^{-2}[(|\eta_b\varphi_-(s) - \varphi_+(s)|\,\|\varphi_- - \eta_a\varphi_+\|_a^s)^2 + (|\varphi_-(s) - \eta_a\varphi_+(s)|\,\|\eta_b\varphi_- - \varphi_+\|_s^b)^2]\,. \tag{8.7}$$

It then follows from (8.3), (8.4) that there exists a constant C such that

$$g_{ab}^2(s) \leqq |1 - \eta_a\eta_b|^{-2}\, C[(\hat\varphi_+(s)\,\|\hat\varphi_-\|_a^s)^2 + (\hat\varphi_-(s)\,\|\hat\varphi^+\|_s^b)^2]$$

Since $|\eta_a\eta_b| \leqq \frac{1}{2}$ on $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$, the conclusion of Lemma 2 is therefore a consequence of the hypothesis (8.1).

The Schwarz inequality for $\mathfrak{F}_{ab}$ yields

$$\begin{aligned} |y_{ab}(s) - (\lambda - l_0)G_{ab}x(s)| &= |G_{ab}[(\mu_{ab} - l_0)y_{ab}(s) - (\lambda - l_0)x(s)]| \\ &\leqq g_{ab}(s)(|\mu_{ab} - l_0|\,\|y_{ab} - x\|_a^b + |\mu_{ab} - \lambda|\,\|x\|)\,. \end{aligned}$$

Hence Lemma 2 and Theorem 4 show that there exists C such that

$$|y_{ab}(s) - (\lambda - l_0)G_{ab}x(s)| \leqq C(|\theta_a| + |\theta_b|)\,, \tag{8.8}$$

$a \leqq s \leqq b$ whenever $a \leqq a_0$, $b_0 \leqq b$.

The solution $f(s)$ of the boundary value problem $L_0 f = 0$, $U_a f = U_a x$, $U_b f = U_b x$ is given by (7.2) or (7.3). The function F defined by

$$F(s) = (\lambda - l_0)G_{ab}x(s) - x(s) + f(s)$$

satisfies $L_0F = 0$, $U_aF = U_bF = 0$, and hence F is the zero function on $a \leqq s \leqq b$. The following uniform estimate is then an immediate consequence of (8.8):

$$\begin{aligned} &y_{ab}(s) = x(s) - f(s) + O(\theta_a) + O(\theta_b)\,, \\ &a \leqq s \leqq b,\ \omega_- < a \leqq a_0,\ b_0 \leqq b < \omega_+\,. \end{aligned} \tag{8.9}$$

THEOREM 5. *If both singularities ω_- and ω_+ of L are of the limit point type, under the assumptions* (6.9)-(6.11), (8.1), *the perturbed characteristic function y_{ab} associated with the characteristic value μ_{ab} of Theorem* 4 *has the uniform representation* (8.9).

9. Asymptotic estimates in the mixed case. In this section, ω_- is supposed to be a limit circle singularity and ω_+ a limit point singularity. The basic problem is (6.16) and the assumptions are (6.10), the second of (6.11), and the first of (2.4).

Proceeding as in § 7, we obtain the representation (7.3) and the inequality below (7.3) where $\varphi_\pm$ are described by (6.15) in the mixed case. According to (6.12), the following relations hold in connection with the limit point singularity ω_+: $\eta_b = o(1)$, $\theta_b^* = o(1)$, and $U_bx/U_b\varphi_- = O(\theta_b) = o(1)$ as $b \to \omega_+$. Since ω_- is a limit circle singularity, it is a consequence of (3.6) that

(9.1) $\theta_a^* = (U_a\varphi_-/U_a\varphi_+)\,||\varphi_+||_a = o(1);\ \eta_a = o(1)$ as $a \to \omega_-$.

In addition to (6.3)-(6.8) we shall use the notation

(9.2) $$\rho_a = U_a x\ .$$

It follows from the postulated end condition $[x\varphi_-](-) = 0$ that for $x \in \mathfrak{D}^1$, $\rho_a = [x\varphi_-](a)[1 + o(1)] = o(1)$ as $a \to \omega_-$, and from (3.6),

$$\theta_a = (U_a x/U_a\varphi_+)\,||\varphi_+||_a = O(\rho_a),\ \omega_- < a \leqq a\ .$$

The analogue of (7.4) in the mixed case is therefore

$$||f||_a^b \leqq C(|\rho_a| + |\theta_b|)\ .$$

The proof of the following theorem is then identical with that of Theorem 4.

THEOREM 6. *If ω_- is a limit circle singularity and ω_+ is a limit point singularity of L, then under the assumptions* (6.10), (6.11), *and the first of* (2.4), *for every λ of the mixed problem* (6.16) *there exists R_0 and a constant C on R_0 such that a unique μ_{ab} of* (2.5) *lies in the interval $|\mu_{ab} - \lambda| \leqq C(|\rho_a| + |\theta_b|)$ whenever $[a, b] \in R_0$. There are normalized characteristic functions x, y_{ab} associated with λ, μ_{ab} respectively such that*

$$||y_{ab} - x|| \leqq C(|\rho_a| + |\theta_b|)\ .$$

10. Uniform estimates in the mixed case. To obtain uniform estimates for characteristic functions on $a \leqq s \leqq b$ in the mixed case, we assume instead of (8.1) that the following are bounded

(10.1) $\varphi_\pm(s)$ on $\omega_- < a \leqq a_0$; $\quad \varphi_+(s)\,||\varphi_-||^s$ on $\omega_- < s < \omega_+$.

Equation (8.7) holds also in the mixed case, and $\eta_a = o(1)$ as $a \to \omega_-$ by (9.1) as well as $\eta_b = o(1)$ as $b \to \omega_+$. Since $||\varphi_+||$ exists by (6.15), there exists a constant C such that

(10.2) $$\begin{aligned} &||\varphi_- - \eta_a\varphi_+||_a^s \leqq C\,||\varphi_-||^s\ , \\ &||\eta_b\varphi_- - \varphi_+||_s^b \leqq C\,|\eta_b|\,||\varphi_-||^b = C\theta_b^* \end{aligned}$$

and $\theta_b^* = o(1)$ as $b \to \omega_+$. Since $\varphi_\pm(s)$ are bounded on $\omega_- < s \leqq b_0$ by (10.1), $g_{ab}(s)$ is bounded on $a \leqq s \leqq b_0$. To show that $g_{ab}(s)$ is bounded also on $b_0 < s \leqq b < \omega_+$, we obtain as in the proof of (8.3), (8.4) that

$$|\eta_a\varphi_+(s)| \leqq C\,|\varphi_-(s)|\ , \qquad |\eta_b\varphi_-(s)| \leqq C\,|\varphi_+(s)|$$

and hence by (8.7), (10.2),

$$g_{ab}^2(s) \leqq |1 - \eta_a\eta_b|^{-2} C_1[(|\varphi_+(s)|\,\|\varphi_-\|^s)^2 + (|\eta_b\varphi_-(s)|\,\|\varphi_-\|^b)^2]$$
$$\leqq |1 - \eta_a\eta_b|^{-2} C_2(|\varphi_+(s)|\,\|\varphi_-\|^s)$$

for some constants C_1, C_2. Then $g_{ab}(s)$ is bounded by the hypothesis (10.1). The following analogue of Theorem 5 is then valid.

THEOREM 7. *If ω_- is a limit circle singularity and ω_+ is a limit point singularity of L, then under the assumptions* (6.10), *the second of* (6.11), *the first of* (2.4), *and* (10.1), *the characteristic function y_{ab} associated with the characteristic value μ_{ab} of Theorem 6 has the following uniform asymptotic representation:*

$$y_{ab}(s) = x(s) - f(s) + O(\rho_a) + O(\theta_h) \tag{10.3}$$
$$a \leqq s \leqq b\,, \qquad \omega_- < a \leqq a_0\,, \qquad b_0 \leqq b < \omega_+$$

where f is given by (7.2).

11. Asymptotic variational formulae for characteristic values. The purpose here is to derive formulae for the change $\mu_{ab} - \lambda$ of characteristic values under the perturbation $\mathfrak{D}^0 \to \mathfrak{D}_{ab}$, valid for a, b in neighborhoods of ω_-, ω_+ respectively.

Let x, y denote the normalized characteristic functions associated with λ, μ as described in Theorems 4 and 5. Let f be the solution (7.3) of the boundary value problem

$$L_0 f = 0\,, \qquad U_a f = U_a x\,, \qquad U_b f = U_b x\,.$$

We conclude from the boundary conditions $U_a y = U_b y = 0$ that $[xy](a) = [fy](a)$ and $[xy](b) = [fy](b)$. Then application of Green's formula

$$(Lx, y)_a^b - (x, Ly)_a^b = [xy](b) - [xy](a)$$

to the differential equations $Lx = \lambda x$, $Ly = \mu y$, and $Lf = l_0 f$ on $[a, b]$ leads to

$$(\lambda - \mu)(x, y)_a^b = (l_0 - \mu)(f, y)_a^b\,; \tag{11.1}$$

$$[fx](b) - [fx](a) = (l_0 - \lambda)(f, x)_a^b\,. \tag{11.2}$$

We obtain as a consequence of Theorem 4 that $\mu = \lambda + o(1)$ and

$$|(x, y)_a^b - (x, x)_a^b| \leqq \|x\|\,\|y - x\|_a^b = o(1)$$

as $a, b \to \omega_-, \omega_+$. Hence

$$(x, y)_a^b = 1 + o(1)\,, \qquad a, b \to \omega_-, \omega_+$$

and (11.1) yields

$$(11.3) \qquad \lambda - \mu = (l_0 - \lambda)(f, y)_a^b[1 + o(1)] .$$

We now appeal to the uniform estimate (8.9) to obtain

$$(f, y)_a^b = (f, x)_a^b - (f, f)_a^b + (\theta_a + \theta_b)(f, 1)_a^b \mathfrak{O}(1) .$$

The following asymptotic variational formula is then a consequence of (11.2) and (11.3):

$$\lambda - \mu_{ab} = [fx](b) - [fx](a) + (l_0 - \lambda)(f, f)_a^b + (\theta_a + \theta_b)(f, 1)_a^b O(1) .$$

In various problems of practical interest (see [5], [6] for detailed references) the first two terms on the right dominate the other terms, and the asymptotic relation

$$(11.4) \qquad \lambda - \mu_{ab} \sim [fx](b) - [fx](a)$$

is valid for $a, b \to \omega_-, \omega_+$. In some cases, $\lambda = 0$ is not a characteristic value and it is permissible to replace l_0 by 0. Then f can be taken as a real valued solution of $Lf = 0$.

EXAMPLE 1. The Hermite operator L given by $Lx = -x'' + s^2x$ will be considered on the interval $-\infty < s < \infty$. In this example, $k(s) = p(s) = 1$, $q(s) = s^2$, $\omega_- = -\infty$, and $\omega_+ = \infty$. Both singularities are limit point, and the basic problem (6.1) has characteristic values $\lambda^{(n)} = 2n + 1$ and normalized characteristic functions

$$x_n(s) = \pi^{-1/4}2^{-(n+1)/2}(n!)^{-1} \exp(-s^2/2)H_n(s) , \qquad n = 0, 1, \cdots$$

where $H_n(s)$ denotes an Hermite polynomial. The well-known [3] asymptotic behavior of $x_n(s)$ as $s \to \infty$ is

$$(11.5) \qquad x_n(s) \sim \pi^{-1/4}2^{(n+1)/2}(n!)^{-1/2}s^n \exp(-s^2/2) .$$

The perturbed problem to be considered is $Ly = \mu y$, $y(a) = y(b) = 0$. In this case l_0 can be replaced by 0, and the solutions φ_+ and φ_- of $L\varphi = 0$ have the asymptotic behavior

$$\log \varphi_\pm(s) \sim \pm\frac{1}{2}s^2 \qquad \text{as } s \to -\infty ;$$

We then obtain from the representation (7.3) of $f(s)$ that $f'(a) \sim ax(a)$ as $a \to -\infty$. Since $x'(a) \sim -x(a)$, $[xf](a) \sim 2ax^2(a)$. Similarly $[xf](b) \sim 2bx^2(b)$. Then (11.4), (11.5) give the asymptotic variational formula

$$\mu_{ab}^{(n)} \sim 2n + 1 + \pi^{-1/2}2^{n+2}(n!)^{-1}[b^{2n+1} \exp(-b^2) - a^{2n+1} \exp(-a^2)]$$
$$a, b \to -\infty, \infty ; \qquad n = 0, 1, 2, \cdots .$$

EXAMPLE 2. Consider the confluent hypergeometric operator L

given by

$$Lx = s\left[-\frac{d^2x}{ds^2} + \frac{x}{4} + \frac{j(j+1)}{s^2}x\right], \qquad 0 < s < \infty$$

in which j is a nonnegative integer. This is related to the Laguerre differential equation, which arises in the quantum mechanical theory of the Hydrogen atom [3]. In this example, $k(s) = 1/s$, $p(s) = 1$, and $q(s) = j(j+1)s^{-2} + 1/4$. The singularity $\omega_+ = \infty$ is in the limit point case, and $\omega_- = 0$ is in the limit point or limit circle case according as $j \geqq 1$ or $j = 0$. If $j = 0$, the singularity is a class 1 limit circle singularity (§ 5) and it can be verified that the variational formula (11.4) is still valid. The basic problem (6.2) has characteristic values $\lambda^{(n)} = n (n \geqq j+1 = 1, 2, \cdots)$ and normalized characteristic functions [3]

$$x_{nj}(s) = -[(n-j-1)!]^{1/2}[(n+j)!]^{-3/2}s^{j+1}e^{-s/2}L_{n+j}^{2j+1}(s) ,$$

where $L_i^h(s)$ denotes the associated Laguerre polynomial, with the asymptotic behaviour

$$x_{nj}(s) \sim (-1)^{n-j-1}[(n+j)!]^{-1/2}[(n-j-1)!]^{-1/2}s^n e^{-s/2} , \qquad s \to \infty ; \tag{11.6}$$

$$x_{nj}(s) \sim [(n+j)!]^{1/2}[(n-j-1)!]^{-1/2}[(2j+1)!]^{-1}s^{j+1} , \qquad s \to 0 . \tag{11.7}$$

The normal solutions of $L\varphi = 0$ have the asymptotic behaviour

$$\log \varphi_\pm(s) \sim \mp\frac{1}{2}s \pm n \log s \qquad (s \to \infty) .$$

For a perturbed problem with boundary operators $U_a x = x(a)$, $U_b x = x(b)$, the representation (7.3) gives $f'(b) \sim x(b)\varphi'_-(b)/\varphi_-(b)$, or $f'(b) \sim \frac{1}{2}x(b)$ as $b \to \infty$. Similarly $f'(a) \sim -jx(a)/a$ as $a \to 0$. Hence

$$[xf](a) \sim -(2j+1)a^{-1}x^2(a) ; \qquad [xf](b) \sim x^2(b) ,$$

and (11.4), (11.6), (11.7) yield the asymptotic formula

$$\mu_{ab}^{(n)} \sim n + \frac{(2j+1)(n+j)!\, a^{2j+1}}{(n-j-1)!\,[(2j+1)!]^2} + \frac{b^{2n}e^{-b}}{(n+j)!\,(n-j-1)!}$$
$$a \to 0 , \quad b \to \infty , \quad j+1 \leqq n = 1, 2, \cdots .$$

To solve the perturbed problem

$$\frac{d^2y}{dS^2} + \left[\frac{2}{S} - \frac{j(j+1)}{S^2} + \nu\right]y = 0 , \qquad y(A) = y(B) = 0 ,$$

we transform the differential equation into the form $Ly = \mu y$ of example 2 by the change of variables

$$S = \mu s/2, A = \mu a/2, B = \mu b/2, \nu = -1/\mu^2$$

and obtain the result

$$\nu_{AB}^{(n)} + \frac{1}{n^2} \sim \frac{2}{n^3}(\mu_{ab}^{(n)} - n) \qquad (A \to 0; B \to \infty) .$$

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