E^3 MODULO A 3-CELL

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If A is a compact continuum in E^n, then E^n/A is the decomposition of E^n whose only nondegenerate element is A. If C is an n-cell in E^n, let $N(C)$ be the set of points on BdC at which BdC is not locally polyhedral.

In [1], Andrews and Curtis proved that if A is an arc in E^n, then $E^n/A \times E^1$ is homeomorphic to E^{n+1}. In Theorem 2 of this paper it is proved that if C is a 3-cell in E^3 such that there exists an arc A on BdC containing $N(C)$, then E^3/A is homeomorphic to E^3/C. It follows that $E^3/C \times E^1$ is homeomorphic to E^4.

J denotes the set of all positive integers and d is the usual metric for E^3. An *n-manifold* is a separable metric space K such that each point of K has a neighborhood which is homeomorphic to E^n. An *n-manifold-with-boundary* is a separable metric space M such that each point of M lies in an open set V such that the closure of V is an n-cell (the homeomorphic image of $\{(x_1, x_2, \cdots, x_n): x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1\}$). If M is an n-manifold-with-boundary, then the *boundary of* M is the set of points of M which do not have neighborhoods homeomorphic to E^n. The boundary of M is denoted by BdM.

The term "interior" is used in two different ways. The *interior* of an n-manifold-with-boundary M is $M - BdM$. If T is a compact connected 2-manifold in E^3 such that $E^3 - T$ is the union of two disjoint open sets each having T as its boundary, then the *interior* of T is the bounded component of $E^3 - T$. In either case the interior of a set L is denoted by (int L). The *exterior of* T is the unbounded component of $E^3 - T$ and is denoted by (ext T). If X is a set in E^3 and e is a positive number, let $Cl(X)$ be the closure of X and $V(X, e)$ be $\{y: y \in E^3 \text{ and } d(X, y) < e\}$.

THEOREM 1. *Let C and A be compact sets in E^3 such that there exist sequences U and V of open sets in E^3 and a sequence h of homeomorphisms of E^3 onto itself such that*

(1) $Cl(U_{i+1}) \subset U_i$, $\cap \{U_j: j \in J\} = C$, U_1 *is bounded,*

(2) $Cl(V_{i+1}) \subset V_i$, $\cap \{V_j: j \in J\} = A$, V_1 *is bounded, and*

(3) $h_i[U_i - Cl(U_{i+1})] = V_i - Cl(V_{i+1})$, *and* $h_i = h_{i-1}$ *on* $E^3 - U_i$.

Then E^3/C is homeomorphic to E^3/A.

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Proof. If $x \in (E^3 - C)$, let $g(\{x\})$ be $\{\lim h_i(x)\}$, and let $g(C)$ be A. Then g is a homeomorphism of E^3/C onto E^3/A.

THEOREM 2. *Let C be a 3-cell in E^3 such that there exists an arc A on BdC such that $N(C) \subset A$. Then E^3/C is homeomorphic to E^3/A.*

Proof. Let C and A satisfy the hypothesis of Theorem 2.

LEMMA 1. *If e is a positive number, there exist a 3-manifold-with-boundary S and a homeomorphism h_e of E^3 onto itself such that* (1) $C \subset (\text{int } S)$, (2) *if* $x \in [E^3 - V(C, e)] \cup A, h_e(x) = x$, *and* (3) $h_e[Cl(\text{int } S)] \subset V(A, e)$.

Proof of Lemma 1. Let P be the solid parallelepiped with the set of vertices

$$\{((-1)^n, (-1)^m, 0): m, n \in J\} \cup \{((-1)^n, (-1)^m, -1): m, n \in J\} .$$

There exists a homeomorphism g of C onto P such that $g[A] = \{(x, 0, 0): -1 \leqq x \leqq 1\}$. There exists a number b, $0 < b < 1$, such that $\{(x, y, z): y^2 + z^2 \leqq b^2 \text{ and } (x, y, z) \in P\} \subset g[V(A, e)]$. Let E be $\{(x, y, z): y^2 + z^2 = b^2 \text{ and } (x, y, z) \in P\}$.

Let D be $g^{-1}[E]$, D_1 be the component of $BdC - D$ containing A, and D_2 be $BdC - Cl(D_1)$. Notice that each of $D \cup D_1$ and $D \cup D_2$ is a 2-sphere which bounds a 3-cell, and $Cl(\text{int } (D \cup D_1)) \subset V(A, e)$.

Now BdD is a simple closed curve which lies on a tame disk, and therefore BdD is a tame simple closed curve. It follows from Theorem 7 of [2] that, without loss of generality, it can be assumed that D is locally polyhedral at each point of (int D). But then D is tame ([3]). Thus it can be assumed that D is a tame disk.

Since D and $Cl(D_2)$ are tame disks which intersect in the boundary of each, $D \cup D_2$ is a tame 2-sphere ([3]). Thus there exists a homeomorphism f of E^3 onto itself such that $f[Cl(\text{int } (D \cup D_2))] = P, f[D] = \{(x, y, 0): (x, y, 0) \in P\}$, and $f[Cl(\text{int } (D \cup D_1)) - D] \subset \{(x, y, z): z > 0\}$. Let U be $f[V(C, e)]$ and W be $f[V(A, e)]$. Since

$$Cl(\text{int } (D \cup D_1)) \subset V(A, e), f[Cl(\text{int } (D \cup D_1))] \subset W .$$

There exists a positive number c such that $Cl(V(P, c)) \subset U$. Let T_0 be $Cl(V(P, c))$. If $x \in (f[C] - T_0)$, let T_x be a polyhedral 3-cell such that $x \in (\text{int } T_x)$ and $T_x \subset (W \cap \{(x, y, z): z > 0\})$. Then there exists a finite subcollection $\{T_1, T_2, \cdots, T_n\}$ of $\{T_x: x \in (f[C] - T_0)\}$

such that $\{T_0, T_1, T_2, \cdots, T_n\}$ covers $f[C]$. Assuming that BdT_0, $BdT_1, \cdots$, and BdT_n are in relative general position, let H be $\cup \{T_i: i = 0, 1, 2, \cdots, n\}$. H is a polyhedral 3-manifold-with-boundary and $f[C] \subset (\text{int } H) \subset H \subset U$. Furthermore, since $(H - \{(x, y, z): z < 0\}) \subset W$ and $H \cap \{(x, y, z): z \leqq 0\}$ is $Cl(V(P, c)) \cap \{(x, y, z): z \leqq 0\}$, there exists a homeomorphism k of E^3 onto itself such that if $x \in (E^3 - U) \cup \{(x, y, z): z \geqq 0\}$, $k(x) = x$, and $k[H] \subset W$.

Let h_e be $f^{-1}kf$ and S be $f^{-1}[H]$. Then h_e and S satisfy the conclusion of Lemma 1.

LEMMA 2. *There exist a sequence $S_1, S_2, \cdots$ of 3-manifolds-with-boundary and a sequence h of homeomorphisms of E^3 onto itself such that*

(1) $S_1 \subset V(C, 1)$,
(2) $S_{i+1} \subset (\text{int } S_i)$,
(3) $\cap \{(\text{int } S_j): j \in J\} = C$,
(4) $\cap \{(\text{int } h_j[S_j]): j \in J\} = A$, *and*
(5) *if* $x \in ((\text{int } S_k) - S_{k+1})$, $h_{k+1}(x) = h_k(x)$.

Proof of Lemma 2. Lemma 2 follows immediately by repeated application of Lemma 1.

For each positive integer i, let U_i be $(\text{int } S_i)$ and V_i be $h_i[(\text{int } S_i)]$. Then the sequences U, V, and h satisfy the hypothesis of Theorem 1. Thus E^3/C is homeomorphic to E^3/A.

COROLLARY 1. *If C satisfies the hypothesis of Theorem* 2, *then $E^3/C \times E^1$ is homeomorphic to E^4.*

COROLLARY 2. *Let C be a* 3-*cell in E^3 such that $N(C)$ is a O-dimensional set. Then $E^3/C \times E^1$ is homeomorphic to E^4.*

Proof. $N(C)$ is a compact O-dimensional set on BdC. Thus there exists an arc A on BdC such that $N(C) \subset A$. Then the result follows from Corollary 1.

THEOREM 3. *Let C be a* 3-*cell in E^3 such that there exists a disk D on BdC containing $N(C)$. Then E^3/C is homeomorphic to E^3/D.*

Proof. The proof of Theorem 3 is analogous to the proof of Theorem 2.

References

1. J. J. Andrews and M. L. Curtis, *n-space modulo an arc*, Ann. of Math., **75** (1962), 1–7.
2. R. H. Bing, *Approximating surfaces by polyhedral ones*, Ann. of Math., **65** (1957), 456–483.
3. E. E. Moise, *Affine structures in 3-manifolds VIII. Invariance of the knot-types; local tame imbedding*, Ann. of Math., **59** (1954), 159–170.

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