ON CERTAIN PROJECTIONS IN SPACES OF CONTINUOUS FUNCTIONS

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1. Introduction. Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of continuous real or complex valued functions on X, with supremum norm. We are concerned with the set $\mathscr{P}$ of positive bounded constant decreasing projections in $C(X)$. That is, $\mathscr{P}$ is the set of bounded linear operators $T: C(X) \rightarrow C(X)$ which have the properties $T^2 = T$, $Tf \geqq 0$ if $f \geqq 0$, $T1 \leqq 1$. A great deal is known about the structure of such T when the range of T is a closed self-adjoint subalgebra of $C(X)$ containing constants [1] [4] [5]. In the present paper we develop a corresponding representation theory for members of $\mathscr{P}$. An application to Markov processes is given.

2. Representation theory. Let X denote the σ-field of Borel subsets of X. We represent the conjugate space of bounded linear functionals on $C(X)$ as the space of regular real or complex Borel measures in X, with variation norm. In all that follows, the topology in $C^*(X)$ will be the $C(X)$ (weak*) topology.

THEOREM 1. *The members of* $\mathscr{P}$ *correspond* $1-1$ *to certain* $C^*(X)$ *valued functions on* X, *as follows. Suppose* $t: X \rightarrow C^*(X)$ *corresponds to* $T \in \mathscr{P}$. *Then* t *and* T *are related by* (i), *and* t *has properties* (ii)–(iv):

(i) $Tf(x) = \int f(x')t_x(dx'),\ x \in X,\ f \in C(X)$

(ii) $t: X \rightarrow C^*(X)$ *is continuous* (*with the* $C(X)$ *topology in* $C^*(X)$).

(iii) $t_x \geqq 0,\ t_x(X) \leqq 1,\ x \in X$

(iv) $t_x = \int t_{x'} t_x(dx'),\ x \in X.$

Proof. Suppose $T \in \mathscr{P}$ is given. Standard representation theory for bounded linear transformations into $C(X)$ gives (i) and (ii) immediately [2, p. 490]. Property (iii) is a consequence of $T \geqq 0$, $T1 \leqq 1$. It is to be noted that the conditions $T \geqq 0$, $T1 \leqq 1$, $T \neq 0$ which characterize the nonzero members of $\mathscr{P}$ are equivalent to the conditions $T \geqq 0$, $\| T \| = 1$. The function t is simply the restriction of the adjoint $T^*: C^*(X) \rightarrow C^*(X)$ to domain X, regarding X as the set of unit point measures in $C^*(X)$. The adjoint itself has the representation

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$$(2) \qquad T^*\lambda = \int t_x \lambda(dx),\ \lambda \in C^*(X)\ ,$$

where the integration is in the weak* sense [3]. (That is, for given $\lambda \in C^*(X)$ the value of the integral in (2) is the element of $C^*(X)$ whose values for $f \in C(X)$ are

$$\int \lambda(dx) \int f(x')t_x(dx'),\ f \in C(X)\ .$$

Condition (iv) is a consequence of $T^2 = T$; the integration again is in the weak* sense. Conversely, any t with properties (ii)–(iv) determines a $T \in \mathscr{P}$ according to (i), and the theorem is proved.

Let φ be the equivalence in X defined by $x_1 \varphi x_2$ if and only if $t_{x_1} = t_{x_2}$. On the quotient space $Y = X/\varphi$ define $\bar{t}: Y \to C^*(X)$ by $\bar{t}_y = t_x$ if $y = \pi x,\ x \in X$ where $\pi: X \to Y$ is the quotient mapping. General considerations show that $\bar{t}$ is a homeomorphism of compact Hausdorff Y and the set $K = \{\bar{t}_y : y \in Y\}$ of various distinct values of t. The quotient mapping is closed, so that the decomposition $\{\pi^{-1}y: y \in Y\}$ of X into closed equivalence classes is upper semicontinuous.

Denote by K_1 the closed convex hull of $K \cup \{0\}$, where 0 is the zero measure. Since $K \cup \{0\}$ is compact, K_1 is compact, and is hence the closed convex hull of its extreme points. Denote by Y_0 the set of all $y \in Y$ such that $\bar{t}_y \neq 0$ is an extreme point of K_1; all extreme points of K_1 are to be found in $\{\bar{t}_y : y \in Y_0\} \cup \{0\}$ [2, p. 440].

THEOREM 2. *For each $y \in Y_0$ the measure $\bar{t}_y$ lives on $\pi^{-1}y$; that is, $\bar{t}_y(E) = \bar{t}_y(E \cap \pi^{-1}y)$, $E \in X$, $y \in Y_0$.* Moreover, $\bar{t}_y(X) = 1$, $y \in Y_0$.

Proof. Property (1.iv) is

$$(3) \qquad \bar{t}_y = \int \bar{t}_{\pi x} \bar{t}_y(dx),\ y \in Y\ ,$$

in terms of $\bar{t}$. Fix $y \in Y_0$, and suppose there exists a closed set F disjoint from $\pi^{-1}y$ such that $\bar{t}_y(F) > 0$. Since $\bar{t}$ is one-to-one and continuous, $\bar{t}\pi F = \{\bar{t}_{\pi x} : x \in F\}$ is a compact set which does not contain $\bar{t}_y$. The closed convex hull of $\bar{t}\pi F$ does not contain $\bar{t}_y$, either (otherwise $\bar{t}_y \in \bar{t}\pi F$, $\bar{t}_y$ being extreme [2, p. 440]). Thus there exists $f \in C(X)$ which separates $\bar{t}_y$ and $\bar{t}\pi F$ strictly. Expressing (3) as

$$\bar{t}_y = \bar{t}_y(F) \int_F \bar{t}_{\pi x} \frac{\bar{t}_y(dx)}{\bar{t}_y(F)} + \int_{X-F} \bar{t}_{\pi x} \bar{t}_y(dx) + [1 - \bar{t}_y(X)]0\ ,$$

we see that $\bar{t}_y$ is expressed as a proper convex combination of elements of K_1 distinct from $\bar{t}_y$. This contradicts the assumption that $\bar{t}_y$ is an extreme point of K_1. The regularity of each $\bar{t}_y$ shows that $\bar{t}_y$ lives on $\pi^{-1}y$ when $y \in Y_0$. The same sort of argument shows that if $\bar{t}_y(X) \neq 0$, then $\bar{t}_y$ is not an extreme point of K_1 unless $\bar{t}_y(X) = 1$.

THEOREM 3. *Y_0 is closed.*

Proof. Define u: $Y \to C^*(Y)$ by $u_y(E) = \bar{t}_y(\pi^{-1}E)$, $E \in \boldsymbol{Y}$, $y \in Y$. The continuity of $\bar{t}$ implies that u is continuous with the $C(Y)$ topology in $C^*(Y)$. From Theorem 2, u_y is for each $y \in Y_0$ the unit point measure at y. Thus for each $f \in C(Y)$ we have

$$f(y) = \int f(y')u_y(dy'),\ y \in Y_0 . \tag{4}$$

Since for each $f \in C(Y)$ the members of (4) are continuous in y, the equality (4) persists for $y \in \bar{Y}_0$. This implies that u_y is the unit point measure at y for each $y \in \bar{Y}_0$. It follows that $\bar{t}_y$ lives on $\pi^{-1}y$ for each $y \in \bar{Y}_0$. It should be clear that each such $\bar{t}_y$, $y \in \bar{Y}_0$, is necessarily an extreme point of K_1, and the theorem follows.

THEOREM 4. *For each $y \in Y$ the measure $\bar{t}_y$ lives on $\pi^{-1}Y_0$; that is, $\bar{t}_y(E) = \bar{t}_y(E \cap \pi^{-1}Y_0)$, $E \in \boldsymbol{X}$, $y \in Y$.*

Proof. Since $\{\bar{t}_y, y \in Y\}$ is in the closed convex hull of compact $\{\bar{t}_y, y \in Y_0\} \cup \{0\}$, for each $y \in Y$ there exists a Borel measure $\nu_y \geqq 0$ on compact Y_0 such that

$$\bar{t}_y = \int_{Y_0} \bar{t}_{y'}\nu_y(dy') \tag{5}$$

in the weak* sense. Let F be an arbitrary closed subset of $X - \pi^{-1}Y_0$, and let $f \in C(X)$ satisfy $f(F) = 1$, $f(\pi^{-1}Y_0) = 0$, $0 \leqq f \leqq 1$. From (5) and Theorem 2 one has $\int f(x)\bar{t}_y(dx) = 0$, $y \in Y$, and hence $\bar{t}_y(F) = 0$, $y \in Y$. Since each $\bar{t}_y$ is regular, the theorem follows.

3. **Invariant measures and functions.** We now characterize the ranges of T^* and T. From (2), any invariant measure $T^*\lambda$ is contained in the weak* closed subspace spanned by $\{t_x, x \in X\}$. From (1.iv), each t_x is invariant, $x \in X$. Thus range (T^*) is the weak* closed subspace spanned by $\{t_x, x \in X\}$. The extreme points $\{\bar{t}_y, y \in Y_0\}$ constitute a minimal spanning set, clearly.

From (1.i), any invariant function Tf is constant on equivalence classes, and so determines an element of $C(Y)$. Restriction of domain to Y_0 gives an element of $C(Y_0)$. Conversely, let f_0 be an arbitrary element of $C(Y_0)$. Define function f by

$$f(x) = \int_{\pi^{-1}Y_0} f_0(\pi x')t_x(dx'),\ x \in X . \tag{6}$$

It follows from Theorem 3 and the Tietze extension theorem that $f \in C(X)$ and hence that $Tf = f$. From Theorem 2, the contraction procedure described above applied to f gives f_0 back again. It should then be clear that (6) establishes an isometric order isomorphism of $C(Y_0)$ and range (T). The isomorphism is algebraic if and only if $Y_0 = Y$ [4].

4. Application to Markov chains. Let $(X_1, \mathscr{F})$ be a measurable space, and let $p(x, E)$, $x \in X_1$, $E \in \mathscr{F}$, be a transition subprobability. That is, $p(x, \cdot)$ is a measure on $\mathscr{F}$ for each $x \in X_1$ and $0 \leqq p(\cdot, E) \leqq 1$ is a measurable function for each $E \in \mathscr{F}$. Denote by $B(X_1, \mathscr{F})$ the Banach space of all bounded real or complex measurable functions on X_1, with supremum norm. Then $P: B(X_1, \mathscr{F}) \to B(X_1, \mathscr{F})$ defined by

$$Pf(x) = \int f(x')p(x, dx'), \quad x \in X_1, \ f \in B(X_1, \mathscr{F}) ,$$

has the properties $P \geqq 0$, $\|P\| \leqq 1$. Suppose there is an operator T (necessarily unique) in the closed convex hull of $\{P^n, n = 1, 2, \cdots\}$ in the weak operator topology with the properties $TP = PT = T$. Then T has the properties $T \geqq 0$, $\|T\| \leqq 1$, and is the projection onto the subspace of invariant functions of P.

We assume without essential loss of generality that $B(X_1, \mathscr{F})$ separates the points of X_1. Then there is a totally disconnected compact Hausdorff space X containing X_1 as a dense subset such that each element of $B(X_1, \mathscr{F})$ extends uniquely to an element of $C(X)$ [2, p. 276]. Operator P becomes an operator $P: C(X) \to C(X)$ with the properties $P \geqq 0$, $\|P\| \leqq 1$. Such an operator necessarily has the form

$$Pf(x) = \int f(x')p_x(dx'), \quad x \in X, \ f \in C(X) ,$$

where $p: X \to C^*(X)$ is continuous with the $C(X)$ topology in $C^*(X)$ and has the properties $p_x \geqq 0$, $p_x(X) \leqq 1$, $x \in X$. Clearly, p is the extension of the given transition subprobability to all of X.

The operator T becomes a projection in $C(X)$ to which our results apply. Each set $\pi^{-1}y, y \in Y_0$ is an ergodic set and $X - \pi^{-1}Y_0$ is the dissipative set, according to

THEOREM 5. *If $y \in Y_0$ then for almost all $x \in \pi^{-1}y$ with respect to $\bar{t}_y$ the measure p_x lives on $\pi^{-1}y$.*

Proof. From $TP = T$ we obtain

$$\int \bar{t}_y(dx) \int p_x(dx')f(dx') = \int \bar{t}_y(x')f(x'), \quad y \in Y, \ f \in C(X) .$$

Fix $y \in Y_0$ and let F be any closed set disjoint from $\pi^{-1}y$. Let $f \in C(X)$ be such that $f(F) = 1$, $f(\pi^{-1}y) = 0$, $0 \leqq f \leqq 1$. The right-hand side above vanishes, from Theorem 2, which requires $p_x(F) = 0$ for almost all x with respect to $\bar{t}_y$. Since p_x is regular, the theorem follows.

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