SIMPLE PATHS ON POLYHEDRA

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In Euclidean d-space ($d \geqq 3$) consider a convex polytope whose $n\,(n \geqq d+1)$ vertices do not lie in a $(d-1)$-space. By the "path length" of such a polytope is meant the maximum number of its vertices which can be included in any single simple path, i.e., a path along its edges which does not pass through any given vertex more than once. Let $p(n, d)$ denote the minimum path length of all such polytopes of n vertices in d-space. Brown [**1**] has shown that $p(n, 3) \leqq (2n + 13)/3$ and Grünbaum and Motzkin [**3**] have shown that $p(n, d) < 2(d-2)n^{\alpha}$ for some $\alpha < 1$, e.g., $\alpha = 1 - 2^{-19}$ and they have indicated how this last value may be improved to $\alpha = 1 - 2^{-16}$. The main object of this note is to derive the following result which, for sufficiently large values of n, represents an improvement upon the previously published bounds.

THEOREM.
$$p(n, d) < (2d+3)((1 - 2/(d+1))n - (d-2))^{\log 2/\log d} - 1 < 3d\, n^{\log 2/\log d}.$$

When $d = 3$ the example we construct to imply our bound is built upon a tetrahedron which we denote by G_0. Its 4 vertices, which will be called the 0th stage vertices, can all be included in a single simple path. Upon each of the 4 triangular faces of G_0 erect a pyramid in such a way that the resulting solid, G_1, is a convex polyhedron with 12 triangular faces. This introduces 4 more vertices, the 1st stage vertices, which can be included in a single simple path involving all 8 vertices of G_1. We may observe that it is impossible for a path to go from a 1st stage vertex to another 1st stage vertex without first passing through a 0th stage vertex.

The convex polyhedron G_2 is formed by erecting pyramids upon all the faces of G_1. Of the 12 2nd stage vertices thus introduced at most 9 can be included in any single simple path since, as before, no path can join two 2nd stage vertices without passing through an intermediate vertex of a lower stage and there are only 8 such vertices available.

The procedure continues as follows: the convex polyhedron G_k, $k \geqq 2$, is formed by erecting pyramids upon the 4.3^{k-1} triangular faces of G_{k-1}. Making repeated use of the fact that the method of construction makes it impossible for a path to join two vertices of the jth stage, $j \geqq 2$, without first passing through at least one vertex of a lower stage we find that at most 9.2^{j-2} of the 4.3^{j-1} vertices of the

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jth stage, $j = 2, 3, \cdots, k$, can be included in a single simple path along the edges of G_k. This and the earlier remarks imply that G_k, $k \geqq 1$, has $2\cdot 3^k + 2$ vertices and at most $9\cdot 2^{k-1} - 1$ of these can be included in a single simple path.

For any integer $n > 4$ let k be the unique integer such that

$$2\cdot 3^k + 2 < n \leqq 2\cdot 3^{k+1} + 2 . \tag{1}$$

Next consider the convex polyhedron with n vertices which can be obtained by erecting pyramids upon $n - (2\cdot 3^k + 2)$ faces of G_k. Then, from considerations similar to those given before, it follows, using (1), that

$$p(n, 3) \leqq 9\cdot 2^k - 1 < 9((n - 2)/2)^{\log 2/\log 3} - 1 . \tag{2}$$

This suffices to complete the proof of the theorem when $d = 3$ since the result is trivially true when $n = 4$.

In the general case the construction starts with a d-dimensional simplex. Upon each of its $(d - 1)$-dimensional faces is formed another d-dimensional simplex by the introduction of a new vertex on the side of the face opposite to the rest of the original simplex in such a way that the resulting polytope is convex. This process is repeated and the rest of the argument is completely analogous to that given for the case $d = 3$. It should be pointed out that the result of Grünbaum and Motzkin holds even for graphs all of whose vertices, but for a bounded number are incident with 3 edges, while in the polytopes described above the distribution of valences is quite different.

In closing we remark that the path length of any 3-dimensional convex polyhedron with n vertices is certainly greater than

$$(\log_2 n/\log_2 \log_2 n) - 1 .$$

Suppose that there exists a vertex, q say, upon which at least $\log_2 n/\log_2 \log_2 n$ edges are incident. Let the vertices at the other ends of these edges be $p_1, p_2, \cdots, p_t$, arranged in counterclockwise order. Each pair, (p_i, p_{i+1}), $i = 1, \cdots, t - 1$, of successive vertices in this sequence determines a unique polygonal face containing the edges $\overline{p_{i+1}q}$ and $\overline{qp_i}$. Traversing this face in a counterclockwise sense gives a path from p_i to p_{i+1} involving at least one edge. Since these faces all lie in different planes it is not difficult to see that these paths may be combined to give a simple path from q to p_1 to p_t whose length is at least $t \geqq \log_2 n/\log_2 \log_2 n$. If there is no vertex upon which this many edges are incident then the required result follows from the type of argument used by Dirac [2; Theorem 5] in showing that the path length is at least of the magnitude of $\log n$ if only a bounded number of edges are incident upon any vertex.

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