ANALYTIC MEASURES

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1. In this note we consider from a measure-theoretic point of view the Helson-Lowdenslager generalization to compact abelian groups of the F. and M. Riesz theorem on analytic measures [3]. Our contribution to this matter is the *proof* of Theorem 1. From Theorem 1 the Helson-Lowdenslager generalization readily follows. That which is new here is the proof of Theorem 1. For the most part, the statement of Theorem 1 can be obtained from the generalization in [3] of the F. and M. Riesz theorem.

We have a second theorem (Theorem 2) which is about analytic measures (Theorem 1 is not) and which adds to the information about analytic measures given in [3]. Although Theorem 2 does not appear in [3] it can be obtained from the generalization in [3] of the F. and M. Riesz theorem, and we will indicate how this may be done at the end of the proof of Theorem 2. In recent work (completed before our work was undertaken) de Leeuw and Glicksberg have found a generalization of the F. and M. Riesz theorem which includes Theorem 2 and much more. Nevertheless, it is hoped that the proof of Theorem 2 given here will be of interest.

Although the proof of Theorem 1 is given in the language of harmonic analysis, we wish to point out that the argument is valid in the more general context of Dirichlet algebras. This however is not true of Theorem 2.

2. Throughout G will denote a compact abelian group with Haar measure σ and with dual group Γ. Following [3], a subset S of Γ is said to be a half-space if S is closed under multiplication and if for each χ in Γ one and only one of the following occurs:

$$\chi = 1$$
$$\chi \in S$$
$$\bar{\chi} \in S .$$

We will assume that Γ contains half-spaces and in all that follows S will denote a fixed half-space in Γ.

$M(G)$ is the space of all regular Borel measures on G. ν in $M(G)$ is said to be analytic (more accurately analytic with respect to S) if the Fourier transform $\hat{\nu}$ vanishes on $\bar{S}$:

$$\hat{\nu}(\chi) = \int \bar{\chi} d\nu = 0$$

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for χ in $\bar{S}$.

$\boldsymbol{A}$ is the algebra of continuous analytic functions on $\boldsymbol{G}$: f belongs to $\boldsymbol{A}$ if and only if f belongs to $\boldsymbol{C}(G)$ and

$$\hat{f}(\chi) = \int \bar{\chi} f d\sigma = 0$$

for χ in $\bar{S}$. A nonnegative measure μ in $\boldsymbol{M}(G)$ with total mass one and such that

$$\int fg d\mu = \left(\int f d\mu\right)\left(\int g d\mu\right)$$

for all f and g in $\boldsymbol{A}$ is called a representing measure for $\boldsymbol{A}$. Among the representing measures is the measure σ.

It is important that the linear space of analytic measures is an A-module: if ν is analytic and f is in A, then $f\nu$ is also analytic.

The classical example of this abstract situation is the case in which G is the circle group, Γ the integer group, and S the positive integers. $\boldsymbol{A}$ is then the algebra of continuous functions on the circle whose Fourier coefficients vanish for negative indices, and the representing measures (other than normalized Lebesgue measure and the unit point masses) are the Poisson kernels μ_r $(0 < r < 1)$ and their translates:

$$\mu_r = P_r \sigma$$
$$P_r(x) = \Sigma r^{|n|} e^{inx} .$$

The celebrated theorem of F. and M. Riesz [4] states: *An analytic measure on the circle is absolutely continuous with respect to Lebesgue measure.*

As usual, $||f||_\infty$ is the supremum norm of f for f in $\boldsymbol{C}(G)$, $|\nu|$ is the total variation of ν and $||\nu||$ the total variation norm of ν for ν in $\boldsymbol{M}(G)$, and $*$ is convolution.

3. THEOREM 1. *Let μ be a representing measure for $\boldsymbol{A}$ and let ν be any measure in $\boldsymbol{M}(G)$. Then there is a sequence f_n in $\boldsymbol{A}$ such that*

$$||f_n||_\infty \leqq 1 \tag{1}$$

$$f_n \to 1 \quad \text{a.e. } \mu \tag{2}$$

$$||f_n \nu - \nu_a|| \to 0 \tag{3}$$

where

$$\nu = \nu_a + \nu_s$$

is the Lebesgue decomposition of ν with respect to μ:

$$\nu_a \ll \mu , \qquad \nu_s \perp \mu .$$

Both the statement and proof of Theorem 1 should be compared with earlier work done by Helson in [2] for the circle group.

Because ν_s is a regular measure singular with respect to μ, we may choose a sequence E_n of compact sets such that

$$|\nu_s|(G \sim E_n) \leqq 1/n \tag{4}$$

$$\mu E_n = 0 \,. \tag{5}$$

Now choose a second sequence F_n of compact sets such that E_n and F_n are disjoint and

$$\mu(G \sim F_n) \leqq 1/n^4 \,. \tag{6}$$

Let v_n be a real continuous function on G such that

$$-2n \leqq v_n \leqq 0 \qquad \text{on } G \tag{7}$$

$$v_n = 0 \qquad \text{on } F_n \tag{8}$$

$$v_n = -2n \qquad \text{on } E_n \tag{9}$$

and let g_n be a real trigonometric polynomial such that

$$-2n \leqq g_n \leqq 0 \qquad \text{on } G \tag{10}$$

$$\| g_n - v_n \|_\infty \leqq 1/n \tag{11}$$

(g_n may be obtained by convolution of v_n with an approximate identity consisting of trigonometric polynomials).

Denote by $\tilde{g}_n$ the trigonometric polynomial conjugate to g_n. Here we mean conjugacy relative to the half-space S: conjugate to the trigonometric polynomial

$$\Sigma a(\chi)\chi$$

is the trigonometric polynomial

$$\Sigma - i\varepsilon(\chi)a(\chi)\chi$$

where

$$\varepsilon(\chi) = \begin{matrix} 1 \text{ if } \chi \in S \\ 0 \text{ if } \chi = 1 \\ -1 \text{ if } \chi \in \bar{S} \,. \end{matrix}$$

Now let

$$k_n = g_n + i\left(\tilde{g}_n - \int \tilde{g}_n d\mu\right) .$$

Then k_n belongs to A, the real part of k_n is g_n, and

$$\int k_n d\mu = \int g_n d\mu . \tag{12}$$

Finally let

$$f_n = e^{k_n} .$$

Then f_n also belongs to A and, because of (10), satisfies (1).

(9) and (11) give

$$|f_n| \leqq e^{-n} \qquad \text{on } E_n$$

and thus

$$\int_{E_n} |f_n| \, d|\nu_s| \leqq e^{-n} ||\nu_s|| .$$

From (1) and (4)

$$\int_{G\sim E_n} |f_n| \, d|\nu_s| \leqq 1/n ,$$

and combining this estimate with the previous estimate leads to

$$||f_n \nu_s|| \to 0. \tag{13}$$

From (8) and (11)

$$\int_{F_n} |g_n|^2 d\mu \leqq 1/n^2 ,$$

and from (6) and (10)

$$\int_{G\sim F_n} |g_n|^2 d\mu \leqq 4/n^2 .$$

Combining these two estimates gives

$$\int |g_n|^2 d\mu \leqq 5/n^2 . \tag{14}$$

Now, since

$$2g_n = k_n + \bar{k}_n ,$$

we have

$$4\int |g_n|^2 d\mu = 2\int |k_n|^2 d\mu + \int k_n^2 d\mu + \int \bar{k}_n^2 d\mu . \tag{15}$$

Moreover, since μ is a representing measure for A and because of (12),

$$\int k_n^2 d\mu = \left(\int g_n d\mu\right)^2 \geqq 0 . \tag{16}$$

(15) and (16) combine to give

$$\int |k_n|^2 d\mu \leqq 2\int |g_n|^2 d\mu \text{ ,}$$ [1]

and this inequality together with (14) shows that the sequence k_n converges to zero in the norm of $L^2(\mu)$. Therefore, passing to a subsequence if necessary, we may assume the sequence k_n converges to zero almost everywhere with respect to μ, and now the sequence f_n satisfies (2).

The conditions (1) and (2) and the dominated convergence theorem imply

$$\|f_n\nu_a - \nu_a\| \to 0 \text{ .} \tag{17}$$

Finally, (13) and (17) give (3).

Because the space of analytic measures is an A-module, statement (3) of Theorem 1 gives the Helson-Lowdenslager theorem on analytic measures [3, Theorem 7]:

COROLLARY 1. *If ν is analytic, then so are ν_a and ν_s.*

Helson and Lowdenslager found more than just the statement of the corollary. They showed [3, Lemma 3]: *If ν is an analytic measure singular with respect to Haar measure, then ν has mean value zero.* This too follows from Theorem 1, but more is true.

THEOREM 2. *Let ν be an analytic measure which is singular with respect to Haar measure. Then $\nu * \mu$ is singular with respect to Haar measure for every representing measure μ.*

Since ν is singular with respect to σ, Theorem 1 (with σ in place of μ) provides a sequence f_n belonging to A such that

$$\|f_n\|_\infty \leqq 1 \tag{18}$$

$$f_n \to 1 \qquad \text{a.e. } \sigma \tag{19}$$

$$\|f_n\nu\| \to 0 \text{ .} \tag{20}$$

Now because ν is analytic and μ is a representing measure for A,

$$(f_n\nu) * \mu = (f_n * \mu)(\nu * \mu) \text{ .} \tag{21}$$

(21) surely holds if ν is replaced by a member of A. But since ν is analytic, ν is in the weak-star closure of $A\sigma$, and since convolution is continuous in the weak-star topology for $M(G)$, (21) continues to hold for ν.

[1] This inequality and its proof are of course not new.

(20) implies

$$\|(f_n\nu)*\mu\| \to 0$$

and so because of (21)

$$\|(f_n*\mu)(\nu*\mu)\| \to 0$$

and this implies, passing to a subsequence if necessary,

$$f_n*\mu \to 0 \qquad \text{a.e. } \nu*\mu\,. \tag{22}$$

On the other hand, (18) and (19) imply, using dominated convergence,

$$\|f_n - 1\| = \int |f_n - 1|\,d\sigma \to 0 \tag{23}$$

and therefore

$$\|(f_n*\mu) - 1\| = \|(f_n - 1)*\mu\| \to 0\,. \tag{24}$$

Because of (24) we may assume, again passing to a subsequence if necessary,

$$f_n*\mu \to 1 \qquad \text{a.e. } \sigma\,. \tag{25}$$

(22) and (25) show that $\nu*\mu$ and σ are carried on disjoint sets, and so they are mutually singular.

We mentioned in the introduction that Theorem 2 can be obtained from the generalization in [3] of the F. and M. Riesz theorem. Indeed, all that is required in our proof of Theorem 2 is a sequence belonging to $\boldsymbol{A}$ and satisfying (20) and (23), and the existence of such a sequence is implied (by using a standard argument) by Lemma 3 and Theorem 7 of [3].

4. Corollary 1 and Theorem 2 applied to the circle group give the F. and M. Riesz theorem. For if ν is an analytic measure on the circle, the singular part with respect to Lebesgue measure, ν_s, is also analytic. But $\nu_s*\mu_r$ is absolutely continuous with respect to Lebesgue measure. Therefore $\nu_s*\mu_r$ is the zero measure, and this implies, as $\hat{\mu}_r$ does not vanish at any point of the integer group, that ν_s is the zero measure.

There is also an F. and M. Riesz theorem for finite Borel measures ν on the real line $\boldsymbol{R}$, which is sometimes proved by mapping a half-plane conformally on the unit disk and using the F. and M. Riesz theorem for the circle. We wish to show that Theorem 1 applied to the Bohr compactification $\boldsymbol{B}$ of the line leads to an easy and, we believe, natural proof of the Riesz theorem for the line.

ν in $\boldsymbol{M}(\boldsymbol{R})$ is said to be analytic if

$$\hat{\nu}(t) = \int_{-\infty}^{\infty} e^{-ist} d\nu(s) = 0$$

for $t < 0$. $\boldsymbol{B}$ is the compact abelian group dual to $\boldsymbol{R}$ when $\boldsymbol{R}$ is given the discrete topology. The mapping π of $\boldsymbol{R}$ into $\boldsymbol{B}$ defined by

$$(\pi s, t) = e^{ist}$$

is a continuous isomorphism of $\boldsymbol{R}$ into $\boldsymbol{B}$, and the image $\boldsymbol{B}_0$ of $\boldsymbol{R}$ is a dense subgroup of $\boldsymbol{B}$. Using the transformation on measures which carries ν in $\boldsymbol{M}(\boldsymbol{R})$ into $\nu\pi^{-1}$ in $\boldsymbol{M}(\boldsymbol{B})$ we may identify $\boldsymbol{M}(\boldsymbol{R})$ with those measures in $\boldsymbol{M}(\boldsymbol{B})$ which are carried on $\boldsymbol{B}_0$. Moreover the Fourier transform of ν in $\boldsymbol{M}(\boldsymbol{R})$ is the same whether we consider ν as an element of $\boldsymbol{M}(\boldsymbol{R})$ or as an element of $\boldsymbol{M}(\boldsymbol{B})$. For $0 < r < 1$, the Cauchy measure μ_r is the measure carried on $\boldsymbol{B}_0$ defined by

$$\hat{\mu}_r(t) = r^{|t|} .$$

Each Cauchy measure is a representing measure for the algebra $\boldsymbol{A}$ of continuous analytic functions on $\boldsymbol{B}$ (here S is the set of positive real numbers), and the Cauchy measures and Lebesgue measure are mutually absolutely continuous.

With this brief description of $\boldsymbol{B}$ it is now easy to show: *An analytic measure on the line is absolutely continuous with respect to Lebesgue measure.*

Assume ν is an analytic measure carried on $\boldsymbol{B}_0$, and denote by σ_0 Lebesgue measure (transferred to $\boldsymbol{B}_0$). Since the Cauchy measures and Lebesgue measure are mutually absolutely continuous, Theorem 1 provides a sequence f_n belonging to $\boldsymbol{A}$ such that

(26) $$\|f_n\|_\infty \leqq 1$$

(27) $$f_n \to 1 \qquad \text{a.e. } \sigma_0$$

(28) $$\|f_n\nu - \nu_a\| \to 0$$

where ν_a is the absolutely continuous component of ν with respect to σ_0.

Consider a Cauchy measure μ_r. Because of (28)

(29) $$\|(f_n\nu)*\mu_r - (\nu_a*\mu_r)\| \to 0 .$$

Also, since ν is analytic,

(30) $$(f_n\nu)*\mu_r = (f_n*\mu_r)(\nu*\mu_r) .$$

Now $f_n*\mu_r$ converges pointwise to 1 on $\boldsymbol{B}_0$, and this is important. This is because of (26) and (27) and because a null set of μ_r remains a null set when translated by an element of $\boldsymbol{B}_0$.

Since ν is carried on $\boldsymbol{B}_0$, $\nu*\mu_r$ is also carried on $\boldsymbol{B}_0$ (and indeed

$\nu * \mu_r \ll \sigma_0$) and therefore, because $f_n * \mu_r$ converges boundedly to 1 on $\boldsymbol{B}_0$,

$$\| (f_n * \mu_r)(\nu * \mu_r) - (\nu * \mu_r) \| \to 0 \ . \tag{31}$$

From (29), (30), and (31) we obtain

$$\nu * \mu_r = \nu_a * \mu_r$$

which implies

$$\nu = \nu_a$$

since $\hat{\mu}_r$ does not vanish at any point of $\boldsymbol{R}$.

5. Corollary 1 and Theorem 2 when applied to the Bohr group give: *If ν is an analytic measure on $\boldsymbol{B}$ and $\nu * \mu_r$ is absolutely continuous with respect to Haar measure (for some $0 < r < 1$), then ν is absolutely continuous.* This is due to Bochner [**1**].

REFERENCES

1. S. Bochner, *Boundary values of analytic functions in several variables and of almost periodic functions*, Ann. of Math., **45** (1944), 708–722.
2. Henry Helson, *On a theorem of F. and M. Riesz*, Colloq. Math., **3** (1955), 113–117.
3. Henry Helson and David Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math., **99** (1958), 165–202.
4. F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès des mathématiciens scandinaves (1916), 27–44.

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