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ISOMETRIC ISOMORPHISMS OF MEASURE ALGEBRAS

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The following theorem is proved:

If G_1 and G_2 are locally compact groups, A_i are algebras of finite regular Borel measures such that $L^1(G_i) \subseteqq A_i \subseteqq \mathscr{M}(G_i)$ for $i = 1, 2$, and T is an isometric algebra isomorphism of A_1 onto A_2, then there exists a homeomorphic isomorphism α of G_1 onto G_2 and a continuous character χ on G_1 such that $T\mu(f) = \mu(\chi(f \circ \alpha))$ for $\mu \in A_1$ and $f \in C_0(G_2)$.

This result was previously known for abelian groups and compact groups (Glicksberg) and when $A_i = L^1(G_i)$ (Wendel) where T is only assumed to be a norm decreasing algebra isomorphism.

A corollary is that a locally compact group is determined by its measure algebra.

If G is a locally compact group with left Haar measure m, then the Banach space $\mathscr{M}(G)$ of finite complex regular Borel measures (the dual of the Banach space $C_0(G)$ of all continuous functions vanishing at infinity on G) can be made into a Banach algebra by defining multiplication of two elements $\mu, \nu \in \mathscr{M}(G)$ to be convolution:

$$\mu * \nu(f) = \iint f(st) d\mu(s) d\nu(t) \qquad \text{for} \qquad f \in C_0(G) .$$

The subspace $L^1(G)$ of all measures absolutely continuous with respect to m is a closed two-sided ideal and hence a subalgebra.

In [1; Theorems 3.1 and 3.2] it is shown that if G_1 and G_2 are either both abelian or both compact, then any algebraic isomorphism T of a subalgebra A_1 of $\mathscr{M}(G_1)$ containing $L^1(G_1)$ onto a subalgebra A_2 of $\mathscr{M}(G_2)$ containing $L^1(G_2)$ which is norm-decreasing on $L^1(G_1)$ has the form

$$(*) \qquad T\mu(f) = \mu(\chi(f \circ \alpha)) \qquad \mu \in A_1 \qquad f \in C_0(G_2)$$

where α is a homeomorphic isomorphism of G_1 onto G_2 and χ is a character on G_1. In this note we shall prove that $(*)$ holds where T is assumed to be an isometry but G_1 and G_2 may be arbitrary locally compact groups. Our starting point will be the theorem of Wendel [2; Theorem 1] that any isometric isomorphism $T: L^1(G_1) \to L^1(G_2)$ is of the form $(*)$.

THEOREM. *If G_1 and G_2 are locally compact groups and T is an isometric isomorphism of a subalgebra A_1 of $\mathscr{M}(G_1)$ containing $L^1(G_1)$*

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onto a subalgebra A_2 *of* $\mathscr{M}(G_2)$ *containing* $L^1(G_2)$ *then* T *has the form* $(*)$. *Conversely, the equation* $(*)$ *defines an isometric isomorphism of* $\mathscr{M}(G_1)$ *onto* $\mathscr{M}(G_2)$ *for every choice of* α *and* χ.

LEMMA.[1] *Let* $\mu, \nu \in \mathscr{M}(G)$. *Then* $\mu \perp \nu$ *if and only if* $\|\mu + \nu\| = \|\mu - \nu\| = \|\mu\| + \|\nu\|$.

Proof. Suppose $\mu \perp \nu$. Then there exists a disjoint partition of G into sets A, B such that $|\mu|(B) = |\nu|(A) = 0$. Thus

$$\begin{aligned}\|\mu \pm \nu\| &= |\mu \pm \nu|(G) = |\mu \pm \nu|(A) + |\mu \pm \nu|(B)\\ &= |\mu|(A) + |\nu|(B) = \|\mu\| + \|\nu\|\,.\end{aligned}$$

Conversely, assme $\|\mu + \nu\| = \|\mu - \nu\| = \|\mu\| + \|\nu\|$. Let $\mu = f\nu + \mu_s$ where $f \in L^1(\nu)$ and $\mu_s \perp \nu$ be the Lebesgue decomposition of μ with respect to ν. Then

$$\begin{aligned}\|\mu \pm \nu\| = \|\mu\| + \|\nu\| &= \|f\nu + \mu_s\| + \|\nu\|\\ &= \|f\nu\| + \|\mu_s\| + \|\nu\|\,.\end{aligned}$$

But $\|\mu \pm \nu\| = \|(1 \pm f)\nu\| + \|\mu_s\|$ so $\|(1 \pm f)\nu\| = \|f\nu\| + \|\nu\|$. Thus $f = 0$ a.e. with respect to ν hence $\mu \perp \nu$.

Proof of theorem. The converse is an easy verification. Let T be an isometric isomorphism of A_1 onto A_2. We shall show first that T maps $L^1(G_1)$ onto $L^1(G_2)$ and hence has the form $(*)$ when restricted to $L^1(G_1)$, and then that $(*)$ extends to all of A_1.

Indeed $L^1(G_i)$ $i = 1, 2$ will be shown to be the intersection of all nontrivial closed left ideals $I \subsetneqq A_i$ which satisfy

$(**)$ $\mu \in I$, $\nu \in A_i$ and $\nu \perp \lambda$ whenever $\mu \perp \lambda$ and $\lambda \in A_i$ imply $\nu \in I$.

T and T^{-1} clearly preserve the property of being a closed left ideal and by the lemma they preserve $(**)$. Thus T maps $L^1(G_1)$ onto $L^1(G_2)$.

Now for $\mu \in L^1(G_i)$, the condition $\nu \in A_i$ and $\nu \perp \lambda$ whenever $\lambda \in A_i$ and $\mu \perp \lambda$ is equivalent to $\nu \ll \mu$. Clearly $\nu \ll \mu$ implies it, and conversely any ν satisfying it must be orthogonal to its singular part λ in its Lebesgue decomposition $\nu = f\mu + \lambda$ with respect to μ since $\lambda \in A_i$. So $L^1(G_i)$ is a closed left ideal satisfying $(**)$. Let $I \subsetneqq A_i$ be any nontrivial closed left ideal satisfying $(**)$. Then I must contain a nonzero L^1 measure since $\alpha * \mu \in L^1$ and is nonzero for $\mu \neq 0$ in I and α is a suitable element in an L^1 approximate identity. The total variation of this measure is absolutely continuous with respect to it, hence in I. By convolving this with an appropriate L^1 approximation to a point

[1] I am indebted to George Reid for suggesting this lemma.

mass, we get a measure $\nu \in I$ strictly positive in a neighborhood of the identity (the convolution of an L^1 and an L^∞ function is continuous). But there is an L^1 approximate identity absolutely continuous with respect to ν, hence in I. Since I is a closed ideal, $L^1 \subseteqq I$.

Thus we have $(*)$ holding for all $\nu \in L^1(G_1)$. Let $\mu \in A_1$, and $\nu \in L^1(G_1)$. Then $\mu * \nu \in L^1(G_1)$ so

$$\begin{aligned}\iint f(\alpha(st))\chi(st)d\mu(s)d\nu(t) &= T(\mu*\nu)(f) = (T\mu * T\nu)(f) \\ &= \iint \chi(t)f(r\alpha t)dT\mu(r)d\nu(t)\end{aligned}$$

so $(*)$ holds for μ and all functions in $C_0(G_2)$ of the form $\int f(r\alpha t)\chi(t)d\nu(t)$ where $f \in C_0(G_2)$ and $\nu \in L^1(G_1)$. This class of functions is dense in $C_0(G_2)$ since ν may be taken in an L^1 approximate identity. Thus $(*)$ holds for all $C_0(G_2)$ by continuity, which proves the theorem.

COROLLARY. *A locally compact group is determined by its measure algebra.*

This corollary was obtained independently by B. E. Johnson (Proc. Amer. Math. Soc. 1964). His results imply the main theorem under the hypothesis that each A_i contains all point masses.

BIBLIOGRAPHY

1. Irving Glicksberg, *Homomorphisms of certain algebras of measures,* Pacific J. Math. **10** (1960), 167-191.

2. J. G. Wendel, *On isometric isomorphism of groups algebras,* Pacific J. Math. **1** (1951), 305-311.

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