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ON THE STRICT AND UNIFORM CONVEXITY OF CERTAIN BANACH SPACES

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Let (X, S, μ) be a σ-finite non-atomic measure space let N be a real valued continuous convex even function defined on the real line such that

(1) $N(u)$ **is nondecreasing for** $u \geqq 0$,

(2) $\lim_{u\to\infty} N(u)/u = \infty$,

(3) $\lim_{u\to 0} N(u)/u = 0$.

Let L_N be the set of all real valued μ-measurable functions f such that $\int_X N(f)d\mu < \infty$. It is known that if there exists a constant k such that $N(2u) \leqq kN(u)$ for all $u \geqq 0$ then L_N is a linear space; in fact, L_N is a B-Space if a norm $||\cdot||$ is defined by setting

$$(*) \qquad ||f|| = \inf\left\{1/\zeta \,\middle|\, \zeta > 0, \int_X N(\eta, f)d\mu \leqq 1\right\}.$$

Denoting the B-space $(L_N, ||\cdot||)$ by L_N^* it is proposed to obtain the necessary and sufficient conditions in order that L_N^* may be (1) Strictly Convex (2) Uniformly Convex.

The linear space L_N admits another norm $|||\cdot|||_{(N)}$ known as the Orlicz norm defined by setting

$$|||f|||_{(N)} = \sup \int_X |fg|\, d\mu$$

for such that $\int_X M(|g|)d\mu \leqq 1$, M being the function complementary to N in the sense of Young. For a discussion of this class of Banach spaces we refer to Mazur and Orlicz [2]. Convexity properties of the Orlicz norm have been studied in Milnes [3].

The space L_N^* may be considered as a modulared linear space defined in Nakano [4]. A nonnegative extended real valued function m defined on a linear space is called a *modular* if

(i) $m(0) = 0$;

(ii) for any $x \in L$ there exists $\xi > 0$ such that $m(\xi x) < \infty$;

(iii) $m(\xi x) = 0$ for all $\xi > 0$ implies $x = 0$;

(iv) $m(x) = \sup_{0\leqq \xi < 1} m(\xi x)$;

(v) m is convex (i.e., $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta = 1, x, y \in L$ imply $m(\alpha x + \beta y) \leqq \alpha m(x) + \beta m(y)$).

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The *modulared linear space* may be considered as a normed linear space if a norm $||\cdot||$ is defined by setting

$$(**) \qquad ||x|| = \inf\{1/\xi \mid \xi > 0 \text{ and } m(\xi x) \leqq 1\} .$$

We note that the linear space L_N is a modulared space if

$$m(f) = \int_X N(f)d\mu ,$$

and the norm $||\cdot||$ defined by (**) is the same as the norm defined in *. In fact, the modulared space L_N is a *finite modulared* space, meaning that $m(f) < \infty$, for all $f \varepsilon L_N$.

A Banach space B is said to be strictly convex if $x, y \in B$, $||x|| = ||y|| = ||(x+y)/2|| = 1$ imply $x = y$. It is uniformly convex if to each ε, $0 < \varepsilon \leqq 2$, there corresponds a $\delta(\varepsilon) > 0$ such that conditions $||x|| = ||y|| = 1$, $||x - y|| \geqq \varepsilon$ imply that $||x + y|| < 2 - \delta(\varepsilon)$.

We shall start by characterizing the strict convexity of L_N^*.

LEMMA 1. *The modulared norm defined in* (**) *associated with a finite modulared space is strictly convex if and only if* $m(x) = m(y) = m\{(x+y)/2)\} = 1$ *imply* $x = y$.

The proof is an easy consequence of the fact that in a finite modulared space, $m(x) = 1$ if and only if $||x|| = 1$ where $||\cdot||$ is the related modulared norm.

THEOREM. *The Banach space* L_N^* *is strictly convex if and only if the N-function N is strictly convex*; *i.e.*,

$$N\left(\frac{u+v}{2}\right) < \frac{1}{2}[N(u) + N(v)]$$

for all real u, v *such that* $u \neq v$.

Proof. Let N be a strictly convex N-function. Let $f, g \in L_N^*$ such that

$$m(f) = m(g) = m\left(\frac{f+g}{2}\right) = 1 .$$

By definition of m it follows that

$$\int_X \left[\frac{N(f) + N(g)}{2}\right] - N\left(\frac{f+g}{2}\right)\Big] d\mu = 0 .$$

whence the convexity of N together with the restrictions on f, and g imply that $f = g$ a.e. Thus by Lemma 1, L_N^* is strictly convex.

To prove the "only if" part, let L_N^* be strictly convex. If possible let N be not strictly convex so that there exist $a, b \geqq 0$ $a \neq b$ such that $N\{(a+b)/2\} = 1/2\,[N(a) + N(b)]$. The continuity of N together with the condition $\lim_{u \to 0} N(u)/u = 0$ imply that N is linear on the interval $[a, b]$ and $a \neq 0$, $b \neq 0$. For $u \in [a, b]$ let $N(u) = pu + q$, where p and q are reals.

Since μ is a nonatomic positive measure there exist pairwise disjoint measurable sets A, B, C of arbitrarily small measure such that

$$\mu(A) = \mu(B) = \mu(C)\ .$$

Let us define functions f, g as follows. Let $f(x) = a$ for $x \in A$, $f(x) = b$ for $x \in B$, and $f(x) = 0$ for all $x \notin A \cup B$. Let $g(x) = b$ for $x \in A$, $g(x) = a$ for $x \in B$, and $g(x) = 0$ for $x \notin A \cup B$, and $g(x) = 0$ for $x \notin A \cup B$. Then

$$m(f) = \int_X N(f)d\mu = [p(a+b) + 2q]\mu(A)\ ,$$

$$m(g) = \int_X N(g)d\mu = [p(a+b) + 2q]\mu(B)\ ,$$

$$m\left(\frac{f+g}{2}\right) = \frac{1}{2}\,[m(f) + m(g)]\ ,$$

and $m(f) = m(g) = m\{(f+g)/2)\}$. By a suitable choice of A, B, C we can assume that

$$m(f) = m(g) = m\left(\frac{f+g}{2}\right) = K < \frac{1}{2}\ .$$

Now let h be a function on X defined by setting

$$h(x) = 0 \text{ if } X \in C\ , \qquad h(x) = t \quad \text{if} \quad x \in C$$

where t is such that $N(t)\mu(C) = 1 - K$. Let $f_1 = h + f$, and $g_1 = h + g$; since $h \wedge f = 0 = h \wedge g$, we obtain

$$m(f_1) = m(h) + m(f) = (1 - K) + K = 1\ .$$

Similarly $m(g_1) = 1$, and further

$$m\left(\frac{f_1 + g_1}{2}\right) = m\left(\frac{f+g}{2} + h\right) = m\left(\frac{f+g}{2}\right) + m(h) = 1\ .$$

Thus we have $f_1 \in L_N^*$, $g_1 \in L_N^*$ and $m(f_1) = m(g_1) = m\{(f_1 + g_1/2)\} = 1$; however $f_1 \neq g_1$. Thus L_N^* is not strictly convex, a contradiction.

We next proceed to characterize the uniform convexity of L_N^*.

It is known [5] that in a modulared semiordered linear space, the modular norm is uniformly convex if and only if the associated norm is uniformly convex. The modulared linear spaces L_N are modulared semiordered linear spaces under the natural pointwise ordering, and the above two norms are respectively the norms $\|\cdot\|_{(N)}$ and $|||\cdot|||_{(N)}$.

With this remark we conclude that the Theorem 8 in Milnes [3] which characterizes the uniform convexity of the norm $|||\cdot|||_{(N)}$ also characterizes the uniform convexity of the norm $\|\cdot\|_{(N)}$.

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