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REFLECTION AND APPROXIMATION BY INTERPOLATION ALONG THE BOUNDARY FOR ANALYTIC FUNCTIONS

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Let there be given a function $f(z)$ analytic in an open connected set, not necessarily simply connected, which is bounded by simple closed analytic curves such that the function is continuous on the closure of the region and such that the real part of the function satisfies boundary conditions that are analytic in a neighborhood of the boundary. We want to interpolate $f(z)$ along the boundaries and find conditions that make the interpolants converge maximally to $f(z)$ throughout the closure of the region. The boundary condition on the real part of $f(z)$ permits the analytic continuation of $f(z)$ across the boundary curves and ensures that we are interpolating at points interior to the region of analyticity. In our error estimates (Theorem 1) maximal convergence depends in an essential way on how far we can reflect $f(z)$ and this in turn depends on the boundary values of the real part of $f(z)$ as well as on the geometry of the given region and its analytic boundaries. In Theorems 2 and 3, a simply connected region is considered. Special points of interpolation are given, these depend only on the parametric representation of the boundary curves and not a conformal map. These points are the image points of the Chebyshev polynomials.

Finally an example is given for a multiply connected region.

As is well known [2] Runge's beautiful theorem shows us that there exist certain "equidistributed" points on the analytic curves such that if we interpolate at these points the interpolants converge to the function. However, the proof depends on knowing the conformal map in order to know what the interpolation points are. Here we shall give conditions that do not require knowledge of the conformal map but for convergence depend on how far we can reflect. Along with these, we shall give simple error estimates. Moreover, we shall show that possible interpolation points are the images on the boundary of roots of the Chebyshev polynomials.

The aspects of this paper which are novel are

(i) the use of reflection

(ii) interpolation at boundary points which are gotten directly from the parametric representation of the boundary and do not depend on a conformal map

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(iii) the use of the images of the roots of the Chebyshev polynomials as possible interpolation points.

Notation. Let R be a connected set whose boundary is Γ. Let $\Gamma = \Gamma^1 \cup \Gamma^2 \cup \cdots \cup \Gamma^s$ where the Γ^j are bounded analytic contours in the $z = x + iy$ plane given by $F^j(x, y) = 0$ with $(F_x^j)^2 + (F_y^j)^2 \neq 0$ along Γ^j, $j = 1, 2, \cdots, s$, where the F^j are real-valued analytic functions of x and y. We assume further that the Γ^j are pairwise disjoint. Let Γ^1 contain in its interior $\Gamma^2, \Gamma^3, \cdots, \Gamma^s$ and contain in its exterior the point at infinity. Let Γ^j contain in its interior a_j $2 \leqq j \leqq s$. As shown in [3] there are "reflection" functions $G_j(z)$ defined on a neighborhood $D^j \cup \Gamma^j \cup \hat{D}^j$ of Γ^j. Assume $G_j(z)$ single-valved on $D^j \cup \Gamma^j \cup \hat{D}^j$ [3] shows.

(1) $z = \overline{G_j(z)}$ is Γ^j.

(2) $G_j(z)$ is analytic on $D^j \cup \Gamma^j \cup \hat{D}^j$, where D^j is contained in the connected R and D^j is contained in the complement of $\Gamma^j \cup D^j$ for $j = 1, 2, \cdots, s$.

(3) The transformation $\hat{z} = \overline{G(z)}$ is an involution; i.e. $\hat{\hat{z}} = z$.

(4) If z is in D^j then $\hat{z}$ is in $\hat{D}^j$ and if z is in $\hat{D}^j$ then $\hat{z}$ is in D^j.

(5) $\overline{G[D^j]} = \hat{D}^j$ and $\overline{G[\hat{D}^j]} = D^j$. We assume the boundary of $\hat{D}^j$, that is not Γ^j, is a contour C^j and $G_j(z)$ is continuous on $\hat{D}^j \cup C^j$.

THEOREM 1. (H 1) *Let $f(z)$ be an analytic single valued function on R whose boundary is Γ such that the real part of $f(z)$ solves the Dirichlet problem in R with real boundary values $B_j(z)$ on Γ^j where $B_j(z)$ are single-valued and continuous in $D^j \cup \Gamma^j \cup \hat{D}^j \cup C^j$ and analytic in $D^j \cup \Gamma^j \cup \hat{D}^j$. Let $f(z)$ be continuous and single-valued on $R \cup \Gamma$.*

(H. 2) *Let $z^j_{n_j1}, z^j_{n_j2}, \cdots, z^j_{n_jn_j+1}$, $n_j = 0, 1, 2, \cdots$, be points of Γ^j, $j = 1, 2, \cdots, s$. Let $p^j_{n_1}(z)$ be the polynomial in z of degree n_1 that agrees with $B_1(z)$ at $z^1_{n_11}, z^1_{n_12}, \cdots, z^1_{n_1n_1+1}$ and let $p^j_{n_j}(z)$ $(2 \leqq j \leqq s)$ be the polynomial in $1/(z - a_j)$ that agrees with $B_j(z)$ for $z - a_j = z^j_{n_j1}, z^j_{n_j2}, \cdots, z^j_{n_jn_j+1}$ where a_j is a point inside c^j.*

(H. 3) *Let*

$$\delta^1_{n_1} = \min_{t \text{ on } C^1} \prod_{k=1}^{n_1+1} |t - z^1_{n_1k}|$$

$$\mu^1_{n_1} = \max_{z \text{ on } \Gamma^1} \prod_{k=1}^{n_1+1} |z - z^1_{n_1k}|$$

and

$$\delta^j_{n_j} = \min_{t \text{ on } C^j} \prod_{k=1}^{n_j+1} \left| \frac{t - a_j - z^j_{n_jk}}{t - a_j} \right|, \quad 2 \leqq j \leqq s$$

$$\mu^j_{n_j} = \max_{z \text{ on } \Gamma^j} \prod_{k=1}^{n_j+1} \left| \frac{z - a_j - z^j_{n_jk}}{z - a_j} \right|, \quad 2 \leqq j \leqq s\,.$$

(H. 4) *Let* $\mu^j_{n_j}/\delta^j_{n_j} \to 0$ *as* $n_j \to \infty$, $j = 1, 2, \cdots, s$. *Then for*

$$\mu = \left(\frac{1}{n_1}, \frac{1}{n_2}, \cdots, \frac{1}{n_s}\right) \quad \textit{and} \quad |\mu| = \max\left\{\frac{1}{n_1}, \frac{1}{n_2}, \cdots, \frac{1}{n_s}\right\}.$$

(C. 1) $R_\mu(z) = \sum_{j=1}^{s} p^j_{n_j}(z)$ *converges uniformly to* $f(z)$ *in* $R \cup \Gamma$ *as* $|\mu| \to 0$ *and thus Re* $R_\mu(z)$ *converges uniformly to* $u(x, y)$ *in* R *and uniformly to* $B_j(z)$ *on* Γ^j.

(C. 2) *Moreover in* $R \cup \Gamma$:

$$|f(z) - R_\mu(z)| \leqq \frac{1}{2\pi} \sum_{j=1}^{s} \frac{L_j M_j}{\delta_j} \mu^j_{n_j}/\delta^j_{n_j}$$

where $L_j =$ *length of* C^j, $M_j = \max\limits_{t \text{ on } C^j} |f(t)|$ *and* $\delta_j = \inf\limits_{z \text{ on } \Gamma^j} \min\limits_{t \text{ on } C^j} |t - z|$.

Proof. In order to avoid notation that only confuses, we shall prove the theorem for the case $s = 2$.

We first analytically continue $f(z)$ into $R \cup \Gamma^1 \cup \hat{D}^1 \cup \Gamma^2 \cup \hat{D}^2$. Let $f^*_j(z) = \overline{f[\overline{G_j(z)}]}$ for z in $\Gamma^j \cup \hat{D}^j$. $f^*_j(z)$ is defined and analytic for z in $\Gamma^j \cup \hat{D}^j$ since $\hat{z} = \overline{G_j(z)}$ is in D^j for z in $\hat{D}^j$ and $G_j(z)$ is analytic for z in $\Gamma^j \cup \hat{D}^j$. But $f^*_j(z) = \overline{f(z)}$ for z on Γ^j, thus on Γ^j

$$f(z) + f^*_j(z) = 2B_j(z) .$$

Thus $f(z) = 2B_j(z) - f^*_j(z)$ analytically continues $f(z)$ into $\Gamma^j \cup \hat{D}^j$ since $f(z)$ is continuous up to and on Γ^j. Moreover, $f(z)$ is continuous on $\Gamma^j \cup \hat{D}^j \cup C^j$ since $G_j(z)$ and $B_j(z)$ are. Thus $f^*_j(z) \equiv \overline{f_j[\overline{G(z)}]}$ analytically continues $f^*(z)$ into $\Gamma^j \cup D^j$ since $f^*_j(z)$ is continuous up to and on Γ^j. Let $\alpha_{n+1}(z) = (z - z^1_{n1})(z - z^1_{n2}) \cdots (z - z^1_{nn+1})$

$$\beta_{m+1}(z) = \left(\frac{1}{z - a_2} - \frac{1}{z^2_{m1}}\right)\left(\frac{1}{z - a_2} - \frac{1}{z^2_{m2}}\right) \cdots \left(\frac{1}{z - a_2} - \frac{1}{z^2_{mm+1}}\right).$$

Then for z on Γ:

$$\begin{aligned} p_{nm}(z) = {} & \frac{1}{2\pi i} \oint_{C^1} \frac{f(t)}{t - z} \frac{\alpha_{n+1}(t) - \alpha_{n+1}(z)}{\alpha_{n+1}(t)} dt \\ & - \frac{1}{2\pi i} \oint_{C^2} \frac{f(t)}{t - z} \frac{\beta_{m+1}(t) - \beta_{m+1}(z)}{\beta_{m+1}(t)} dt \end{aligned}$$

where $p_{nm}(z)$ is a rational function of z, $p_{nm}(z) = p^1_n(z) + p^2_{m+1}(z)$ in which $p^1_n(z)$ is the polynomial in z of degree $\leqq n$ got by interpolating $f(z)$ along Γ^1 at $z^1_{n1}, z^1_{n2}, \cdots, z^1_{nn+1}$ and $p^2_{m+1}(z)$ is the polynomial of degree $m + 1$ in $1/(z - a_2)$ got by interpolating $f(z)$ along Γ^2 so that $p^2_{m+1}(a_2 + z^2_{mj}) = f(z^2_{mj})$. To see the latter let $x = 1/(z - a_2)$ and $y =$

$1/(t-a_2)$ then $\beta_{m+1}(t) = b_{m+1}(y)$ where $b_{m+1}(y)$ is a monic polynomial in y of degree $\leqq m+1$, thus we have

$$\beta_{m+1}(t) - \beta_{m+1}(z) = b_{m+1}(x) - b_{m+1}(y) = (x-y)\sum_{i=1}^{m} a_i(x) y^i$$

where $a_i(x)$ are polynomials in x of degree $\leqq m$. But

$$x - y = \frac{1}{z-a_2} - \frac{1}{t-a_2} = (t-z)xy$$

thus

$$\int_{C^2} \frac{f(t)}{t-z} \frac{\beta_{m+1}(t) - \beta_{m+1}(z)}{\beta_{m+1}(t)}\, dt$$

is a polynomial of degree $\leqq m+1$ in $1/(z-a_2)$. The error for z on Γ is given by:

$$f(z) - p_{nm}(z) = \frac{1}{2\pi i}\int_{C^1} \frac{f(t)}{t-z}\frac{\alpha_{n+1}(z)}{\alpha_{n+1}(t)}\, dt - \frac{1}{2\pi i}\int_{C^2}\frac{f(t)}{t-z}\frac{\beta_{m+1}(z)}{\beta_{m+1}(t)}\, dt\,.$$

Note that:

$$|\alpha_{n+1}(z)| \leqq \mu_n^1,\ |\alpha_{n+1}(t)| \geqq \delta_n^1 \text{ for } z \text{ on } \Gamma^1 \text{ and } t \text{ on } C^1 \text{ and:}$$

$$\frac{\dfrac{1}{z-a_2} - \dfrac{1}{z_{n1}^2}}{\dfrac{1}{t-a_2} - \dfrac{1}{z_{n1}^2}} = \frac{z-a_2-z_{n1}^2}{t-a_2-z_{n1}^2}\,\frac{t-a_2}{z-a_2}$$

and thus

$$\left|\frac{\beta_{m+1}(z)}{\beta_{m+1}(t)}\right| \leqq \frac{\mu_m^2}{\delta_m^2} \text{ for } z \text{ on } \Gamma^2 \text{ and } t \text{ on } C^2.$$

From these it follows:

$$|f(z) - p_{nm}(z)| \leqq \frac{1}{2\pi}\left\{\frac{L_1M_1}{\delta_1}\frac{\mu_n^1}{\delta_1} + \frac{L_2M_2}{\delta_2}\frac{\mu_m^2}{\delta_m^2}\right\} \qquad \text{for } z \text{ on } \Gamma,$$

where L_j is the length of C^j,

$$M_j = \max_{t \text{ on } C^j}|f(t)|\,,\quad \text{and}\quad \delta_j = \inf_{z \text{ on } \Gamma^j}\ \min_{t \text{ on } C^j}|t-z|$$

which is the result.

We next consider the case when Γ is a single analytic contour and ($C^j = C$) we write Γ in parametric form as $z(\sigma) = x(\sigma) + iy(\sigma)$ where $-1 \leqq \sigma \leqq 1$. Let $|z(\sigma_2) - z(\sigma_1)| \leqq A\,|\sigma_2 - \sigma_1|$, let Γ contain the origin and

THEOREM 2 (H. 1) *Let $f(z)$ be an analytic single-valued function on R whose boundary is Γ such that the real part of $f(z)$ solves the Dirichlet problem in R with real boundary values $B(z)$ on Γ where $B(z)$ is a single-valued analytic function on $D \cup \Gamma \cup \hat{D}$ continuous on $\Gamma \cup \hat{D} \cup C$. Let $f(z)$ be continuous and single-valued on $R \cup \Gamma$.*

(H. 2) *Let $z_j^n = z(\sigma_j^n)$ where*

$$\sigma_j^n = \cos [(2j - 1)\pi/(2n + 2)] , \ j = 1, 2, \cdots, n + 1$$

(H. 3) $\delta = \inf\limits_{z \text{ on } \Gamma} \min\limits_{t \text{ on } C} |t - z|$

(H. 4) $A < 2\delta$.

Then

(C. 1) $p_n(z) = \sum\limits_{j=1}^{n+1} f(z_j^n) \dfrac{\omega_{n+1}(z)}{\omega'_{n+1}(z_j^n)(z - z_j^n)}$, *where*

$\omega_{n+1}(z) = \prod_{k=1}^{n+1} (z - z_k^n)$ *and prime denotes differentiation, converges uniformly to $f(z)$ on $R \cup \Gamma$ as $n \to \infty$*

(C. 2) $|f(z) - p_n(z)| \leqq \dfrac{LM}{\pi\delta}\left(\dfrac{A}{2\delta}\right)^{n+1}$

where M is a constant depending on f, L is length of Γ.

Proof. As in the proof of Theorem 1 we have for z on Γ

$$|f(z) - p_n(z)| \leqq \frac{L}{2\pi} M \max_{z \text{ on } \Gamma} |\omega_{n+1}(z)|/\delta^{n+2} .$$

But

$$\begin{aligned} |\omega_{n+1}(z)| &= |(z - z_1^n)(z - z_2^n) \cdots (z - z_{n+1}^n)| \\ &\leqq A^{n+1} |(\sigma - \sigma_1^n)(\sigma - \sigma_2^n) \cdots (\sigma - \sigma_{n+1}^n)| \end{aligned}$$

where the σ_j^n are the roots of the Chebyshev polynomial

$$T_{n+1}(\sigma) = \cos [(n + 1) \text{ arc } \cos \sigma]$$

of degree $n + 1$. Thus since $(\sigma - \sigma_1^n)(\sigma - \sigma_2^n) \cdots (\sigma - \sigma_{n+1}^n)$ is monic

$$|\omega_{n+1}(z)| \leqq A^{n+1} T_{n+1}(\sigma)/2^n .$$

Thus

$$\max_{z \text{ on } \Gamma} |\omega_{n+1}(z)| \leqq A^{n+1}/2^n \quad \text{and} \quad |f(z) - p_n(z)| \leqq \frac{LM}{\delta\pi}\left(\frac{A}{2\delta}\right)^{n+1} .$$

Next let Γ: $z(s) = x(s) + iy(s)$ where s is arc length $0 \leqq s \leqq L$.

THEOREM 3. (H. 1) *Same as Theorem 2.*

(H. 2) *Same as Theorem 2 but $z_j^n = z(s_j^n)$ where*

$$s_j^n = \frac{L}{2}\cos\,[(2j-1)\pi/2(n+1)] + \frac{L}{2}\,,\ j = 1, 2, \cdots, n+1\,.$$

(H 3) $\delta = \inf_{z \text{ on } \Gamma} \min_{t \text{ on } C} |t - z|\,.$

(H 4) $L < 4\delta.$

(C 1) *Same as Theorem* 2.

(C 2) $|f(z) - p_n(z)| \leqq \dfrac{LM}{\pi\delta}\left(\dfrac{L}{4\delta}\right)^{n+1}$

where M *is a constant depending on* f.

Proof. As in the proof of theorem for z on Γ:

$$|f(z) - p_n(z)| \leqq \frac{L}{2\pi} M \max |\omega_{n+1}(z)|/\delta^{n+2}\,.$$

But since $|z - z_j^n| \leqq |s - s_j^n|$ where $z = z(s)$ and $z_j^n = z(s_j^n)$ and since $|(s - s_1^n)(s - s_2^n) \cdots (s - s_{n+1}^n)| \leqq L^{n+1}/2^{2n+1}$ see e.g. [1] we have

$$|f(z) - p_n(z)| \leqq \frac{LM}{\pi\delta}\left(\frac{L}{4\delta}\right)^{n+1}.$$

EXAMPLE. We shall now apply the ideas of this paper to a particular geometrical configuration. Let

Γ^1 be a circle of radius 15 centered at the origin

Γ^2 be a circle of radius 1 centered at $(-13, 0)$

Γ^3 be an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a = 1.075,\ b = 1\,.$$

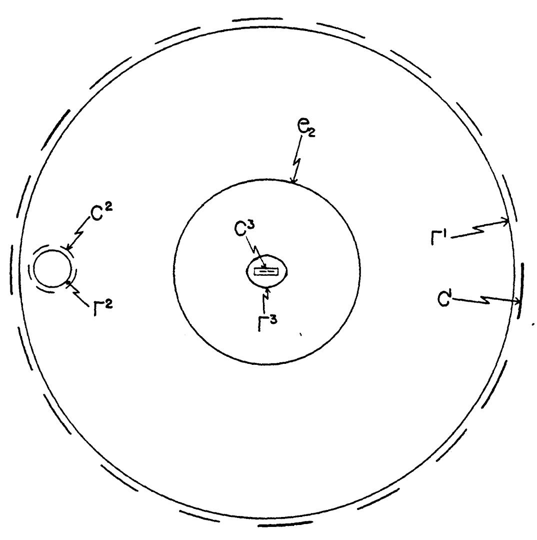

Fig. 1.

Let R be the interior of Γ^1 less Γ^2, Γ^3 and the interiors of Γ^2 and Γ^3. Let $f(z)$ be analytic on R and continuous on $R \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$. Moreover let the real part of $f(z)$ satisfy boundary conditions $B_1(z)$ on Γ^1, $B_2(z)$ on Γ^2 and $B_3(z)$ on Γ^3 where:

$B_1(z)$ is analytic on $|z| = 15$

$B_2(z)$ is analytic on $|z + 13| = 1$

and $B_3(z)$ is analytic in and on $\Gamma^3 - \{-.395 < x < .395, y = 0\}$ See figure.

For example we might have $Re\, f(z) = P_k(x, y)$ on Γ^k $(k = 1, 2, 3)$ where $P_k(x, y)$ is a polynomial.

Then since:

$$G_1(z) = (15)^2 z^{-1}$$
$$G_2(z) = (z + 13)^{-1} - 13$$

(E. 1) $$G_3(z) = \frac{z(a^2 + b^2) \pm 2ab\sqrt{z^2 + b^2 - a^2}}{a^2 - b^2} \quad a > b \text{ see [3]}$$

and $z = \hat{z} = \overline{G_k(z)}$ on Γ_k we have on Γ_k

$$P_k\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = P_k\left[\frac{z + G_k(z)}{2}, \frac{z - G_k(z)}{2i}\right] \equiv B_k(z)$$

which are meromorphic functions that fulfill the requirements of $B_1(z)$, $B_2(z)$ and $B_3(z)$ (in the case of $B_3(z)$ we make a cut between the foci $\pm\sqrt{a^2 - b^2}$).

Let

$$r_k = 15 \exp\left(\frac{2\pi i}{n + 1} k\right) \quad \text{and} \quad \alpha_{n+1}(z) = (z - r_1)(z - r_2) \cdots (z - r_{n+1})$$

and

$$p_n(z) = \sum_{k=1}^{n+1} f(r_k) \frac{\alpha_{n+1}(z)}{\alpha_{n+1}(r_k)(z - r_k)}$$

where the prime signifies differentiation. $p_n(z)$ is clearly the polynomial of degree $\leqq n$ that interpolates $f(z)$ at $z = r_k$ on Γ^1, $k = 1, \cdots, n + 1$. Next let

$$s_k = \exp\left(\frac{2\pi i}{m + 1} k\right)$$

and

$$\beta_{m+1}(z) = \left(\frac{1}{z + 13} - \frac{1}{s_1}\right)\left(\frac{1}{z + 13} - \frac{1}{s_2}\right) \cdots \left(\frac{1}{z + 13} - \frac{1}{s_{m+1}}\right)$$

and

$$q_m(z) = \sum_{k=1}^{m+1} f(s_k - 13) \frac{\beta_{m+1}(z)}{B'_{m+1}(s_k - 13)\left(\frac{1}{s_k} - \frac{1}{z+13}\right)(s_k)^2}$$

$q_m(z)$ is a polynomial of degree $\leqq m$ in $1/(z+13)$ such that $q_m(s_k - 13) = f(s_k - 13)$ where $s_k - 13$ is on Γ^2, $k = 1, 2, \cdots, m+1$.

Finally let l be the length of the ellipse Γ^3 and

$$\sigma_k = \cos[(2k-1)\pi/2(j+1)], k = 1, 2, \cdots, j+1 .$$

Then the ellipse Γ^3 can be written

$$z(\sigma) = x(\sigma) + iy(\sigma) = a\cos(2\pi\sigma/l) + ib\sin(2\pi\sigma/l), \ -l/2 \leqq \sigma \leqq l/2$$

σ is are length parameter shifted. Let

$$t_k = z(\sigma_k l/2) \quad \text{and} \quad \kappa_{j+1}(z) = \left(\frac{1}{z} - \frac{1}{t_1}\right)\left(\frac{1}{z} - \frac{1}{t_2}\right)\cdots\left(\frac{1}{z} - \frac{1}{t_{j+1}}\right)$$

and

$$r_j(z) = \sum_{k=1}^{j+1} f(t_k) \frac{\kappa_{j+1}(z)}{\kappa'_{j+1}(t_k)\left(\frac{1}{t_k} - \frac{1}{z}\right)(t_k)^2}$$

$r_j(z)$ is clearly the polynomial in $1/z$ of degree $\leqq j$ such that $r_j(t_k) = f(t_k)$ $k = 1, 2, \cdots, j+1$ where t_k is on Γ^3.

Then the assertion is

$$p_n(z) + q_m(z) + r_j(z)$$

converges uniformly to $f(z)$ on $R \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ as

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{j} \to 0 .$$

For Γ^1, we use Runge's theorem. Since $B_1(z)$ is analytic on Γ^1, then $f(z)$ can be continued across Γ^1, i.e. $f(z)$ is analytic for $15 \leqq |z| \leqq 15 + \varepsilon$ where ε is some positive number. Thus in the notation of the theorem

$$\begin{aligned}
\delta_n^1 &= \min_{|t|=15+\varepsilon} \left|\prod_{k=1}^{n+1}(t - r_k)\right| = \min_{|t|=15+\varepsilon} |t^{n+1} - 15^{n+1}| \\
&= \min_{|\tau|=1+\varepsilon/15} 15^{n+1}|\tau^{n+1} - 1| \geqq 15^{n+1} \min_{|\tau|=1+\varepsilon/15} \{|\tau|^{n+1} - 1\} \\
&= 15^{n+1}\{[1 + \varepsilon/15]^{n+1} - 1\} \\
\mu_n^1 &= \max_{|z|=15} \left|\prod_{k=1}^{n+1}(z - r_k)\right| = 15^{n+1}\max_{|\zeta|=1}|\zeta^{n+1} - 1| \leqq 2\cdot 15^{n+1}
\end{aligned}$$

and

(E 2) $u_n^1/\delta_n^1 \leqq 2/\{[1+\varepsilon/15]^{n+1}-1\} \to 0$ as $n \to \infty$.

For Γ^2, since $B_2(z)$ is analytic on Γ^2, then $f(z)$ can be continued across Γ^2, i.e. $f(z)$ is analytic for $1-\varepsilon \leqq |z+13| \leqq 1$ where ε is some positive number. Thus if $C^2 = \{z: |z+13| = 1-\varepsilon\}$ we have:

$$\max_{z \text{ on } \Gamma^2} |\beta_{m+1}(z)| = \max_{z \text{ on } \Gamma^2} \left| \prod_{k=1}^{m+1} \left(\frac{1}{z+13} - \frac{1}{s_k} \right) \right| = \max_{z \text{ on } \Gamma^2} \left| \left(\frac{1}{z+13} \right)^{m+1} - 1 \right|$$

$$= \max_{|\zeta|=1} \left| \left(\frac{1}{\zeta} \right)^{m+1} - 1 \right| \leqq 2$$

and

$$\min_{t \text{ on } C^2} |\beta_{m+1}(t)| = \min_{t \text{ on } C^2} \left| \left(\frac{1}{t+13} \right)^{m+1} - 1 \right|$$

$$= \min_{|\zeta|=1-\varepsilon} \left| \left(\frac{1}{\zeta} \right)^{m+1} - 1 \right| \geqq \left(\frac{1}{1-\varepsilon} \right)^{m+1} - 1 .$$

From these we see that

$$\text{(E. 3)} \qquad \frac{\max\limits_{z \text{ on } \Gamma^2} |\beta_{m+1}(z)|}{\min\limits_{t \text{ on } C^2} |\beta_{m+1}(t)|} \leqq \frac{2}{\left(\frac{1}{1-\varepsilon} \right)^{m+1} - 1} \to 0 \text{ as } m \to \infty ,$$

For Γ^3 we note from the reflection function $G(z)$ given by (E. 1) that the interior of the ellipse Γ^3 minus the line $-c \leqq x \leqq c$, $c^2 = a^2 - b^2$, is reflected exterior to the given ellipse but interior to the ellipse e_2

$$\frac{\hat{x}^2}{\hat{a}^2} + \frac{\hat{y}^2}{\hat{b}^2} = 1$$

where $\hat{a} = (a^2+b^2)/c$, $\hat{b} = 2\,ab/c$.

In the case of our ellipse we have $a = 1.075$, $b = 1$ and $c = .395$, $\hat{a} = 5.46$, $\hat{b} = 5.44$, thus e_2 is contained in Γ^1, and does not intersect or contain points of Γ^2 and thus $f(z)$ can be extended to be analytic in $\Gamma^3 - \{z \mid -.395 < x < .395,\ y = 0\}$.

The length of the ellipse Γ^3 is given by:

$$l = 4\mathrm{a} \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta \, d\,\theta}$$

where

$k = c/a < 1$. In our case $k = .368$ and thus
$l = 4a$ (1.516)

using a table for elliptic integrals. Let c^3 be the rectangular contour

$$(-.395 - \varepsilon \leqq x \leqq .395 + \varepsilon, y = -\varepsilon) \ , \ (x = .395 + \varepsilon, -\varepsilon \leqq y \leqq \varepsilon) \ ,$$
$$(-.395 - \varepsilon \leqq x \leqq .395 + \varepsilon, y = \varepsilon) \ , \ (x = -.395 - \varepsilon, -\varepsilon \leqq y \leqq \varepsilon)$$

where $\varepsilon > 0$ is arbitrarily small. Then consider

$$\frac{1}{2\pi i} \int_{\sigma^3} \frac{f(t)}{t - z} \frac{\kappa_{j+1}(z)}{\kappa_{j+1}(t)} \, dt \, .$$

But

$$|\kappa_{j+1}(z)| = \left| \prod_{k=1}^{j+1} \left(\frac{1}{z} - \frac{1}{t_k} \right) \right| = \left| \prod_{k=1}^{j+1} \frac{t_k - z}{t_k z} \right|$$
$$\leqq \prod_{k=1}^{j+1} \frac{\sigma - \sigma_k l/2}{|t_k||z|} \leqq \left(\frac{l}{2} \right)^{j+1} \prod_{k=1}^{j+1} \frac{|\theta - \sigma_k|}{|t_k|} \, . \quad \sigma = \text{arc length},$$

where $-1 \leqq \theta \leqq 1$ since $|z| \geqq 1$ for z on Γ^3. Also for t on C^3

$$|\kappa_{j+1}(t)| = \left| \prod_{k=1}^{j+1} \frac{t_k - t}{t_k t} \right| \geqq (a - c - \eta\varepsilon)^{j+1} \prod_{k=1}^{j+1} \frac{1}{|t_k||t|}$$

where η is some fixed constant. But since $|t| \leqq c + \varepsilon/2$ for t on c^3 we see that

$$|\kappa_{j+1}(t)| \geqq \left(\frac{a - c - \eta\varepsilon}{c + \varepsilon\sqrt{2}} \right)^{j+1} \prod_{k=1}^{j+1} \frac{1}{|t_k|} \, .$$

Combining the above results gives

$$A \equiv |\kappa_{j+1}(z)/\kappa_{j+1}(t)| \leqq \left(\frac{c + \varepsilon\sqrt{2}}{a - c - \eta\varepsilon} \right)^{j+1} \left(\frac{l}{2} \right)^{j+1} 2^{-j} \, |T_{j+1}(\theta)|$$

where we have utilized the fact that the σ_k are the roots of the Chebyshev polynomial $T_{j+1}(\theta) = \operatorname{cox} [(j + 1) \text{ are } \cos \theta]$. Thus

$$A \leqq 2 \left(\frac{c + \varepsilon\sqrt{2}}{a - c - \eta\varepsilon} \right)^{j+1} \left(\frac{l}{4} \right)^{j+1} .$$

But

$$\frac{l}{4a} = 1.516 < \frac{1}{a} \frac{a - c - \eta\varepsilon}{c + \varepsilon\sqrt{2}} = \frac{1}{1.075} \frac{.680 - \eta\varepsilon}{.395 + \sqrt{2\varepsilon}} \, , = 1.60 + g(\varepsilon)$$

where $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus for ε sufficiently small

$$l/4 \leqq \frac{a - c - \eta\varepsilon}{c + \varepsilon\sqrt{2}} \, . \tag{E. 4}$$

Utilizing (E. 2), (E. 3) and (E. 4) we have from Theorem 1 that $p_n(z) + p_m(z) + r_j(z)$ converges uniformly to $f(z)$ in $R \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ as $(1/n) + (1/m) + (1/j) \to 0$.

We remark finally that there would be no new difficulties if Γ had contained in addition $\Gamma^4 \cup \Gamma^5 \cup \Gamma^6$ where Γ^4 is the circle $|z - 10i| = 4$, Γ^5 the circle $|z + 10\,i| = 4$ and Γ^6 is the circle $|z - 12| = 2$.

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