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STABILITY OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS IN HILBERT SPACE

GERT ALMKVIST

In this paper we study the stability of the solutions of the differential equation

$$u'(t) = A(t) \cdot u(t) \tag{1}$$

for $t \geqq 0$ in a separable Hilbert space. It is assumed that $A(t)$ is periodic with period one and satisfies the following symmetry condition: There exists a continuous constant invertible operator Q such that

$$A(t)^* = -Q \cdot A(t) \cdot Q^{-1} \qquad \text{for all} \quad t \geqq 0 \,.$$

We use a perturbation technique. Let $A(t) = A_0(t)+B(t)$ where $A_0(t)$ is compact and antihermitian for all t. We denote by $U_0(t)$ the solution operator of $u'(t) = A_0(t)u(t)$. It is shown that (1) is stable if $B(t)$ satisfies a certain smallness condition involving the distribution of the eigenvalues of $U_0(1)$ and the action of $B(t)$ on the eigenvectors of $U_0(1)$. The results can be applied to the second order equation

$$y'' + C(t)y = 0$$

where $C(t)$ is selfadjoint for all t.

Throughout this paper we consider the differential equation (1) where u is a function from the positive reals, $\mathbf{R}^+$, into a separable Hilbert space X with norm $\|x\| = (x, x)^{1/2}$. A is a function from $\mathbf{R}^+$ into $B(X)$, the algebra of continuous linear operators on X. We assume that $A(t)$ is Bochner integrable on every finite subinterval of $\mathbf{R}^+$. Then for a given initial value $u(0)$, there exists a unique solution of (1) (see [4, p. 521]).

Further we always assume that $A(t)$ is periodic. It is no restriction to assume that the period is one, that is $A(t+1) = A(t)$ for all $t \in \mathbf{R}^+$.

The equation (1) is said to be *stable* if for every initial value $u(0)$, there exists a constant M, such that $\|u(t)\| \leqq M$ for all $t \in \mathbf{R}^+$. It is convenient to study the equation

$$U(t)' = A(t)U(t) \,, \qquad U(0) = I \tag{2}$$

in $B(X)$. Using the principle of uniform boundedness it is easily seen that (1) is stable if and only if the solution of (2) is bounded.

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Let $$\Phi(A) = \lim_{\alpha \to +0} \alpha^{-1}(\| I + \alpha A \| - 1)$$

denote the Gateau differential of A. When X is a Hilbert space $\Phi(A)$ can be calculated by the formula $\Phi(A) = \sup_{\|x\|=1} Re(Ax, x)$

PROPOSITION 1. If $\int_0^1 \Phi(A(t))dt \leqq 0$, then (1) is stable.

Proof. Let n be the greatest integer $\leqq t$. Then using [1, Th. 4] we get

$$\| U(t) \| \leqq \exp \int_0^t \Phi(A(s))ds \leqq \exp \left(n\int_0^1 \phi(A(s))ds\right) \cdot \exp \int_0^{t-n} \phi(A(s))ds$$
$$\leqq \exp \int_0^1 | \Phi(A(s)) | \, ds$$

which ends the proof.

From now on we assume that $A(t)$ satisfies the following symmetry condition:

There exists a constant continuous operator Q such that Q^{-1} is continuous and

$$\text{(S)} \qquad A(t)^* = - QA(t)Q^{-1} \qquad \text{for all} \quad t \geqq 0 .$$

Here A^* denotes the adjoint of A.

PROPOSITION 2. Condition (S) is equivalent to

$$U(t)^* = QU(t)^{-1}Q^{-1} \qquad \text{for all} \quad t \geqq 0 .$$

Proof. We have $U^*(0)QU(0) = Q$ because $U(0) = I$. But

$$\frac{d}{dt}(U(t)^*QU(t)) = U(t)^*A^*(t)QU(t) + U(t)^*QA(t)U(t) = 0$$

if and only if

$$A^*(t)Q + QA(t) = 0 .$$

Let $\sigma(U)$ be the spectrum of U. From Proposition 2 it follows that $\sigma(U^*(t)) = \sigma(QU^{-1}(t)Q^{-1}) = \sigma(U^{-1}(t)$ that is $\lambda \in \sigma(U(t))$ implies $\bar{\lambda}^{-1} \in \sigma(U(t))$.

PROPOSITION 3. If Q is positive definite, then (1) is stable.

Proof. Q has a positive definite square root S, that is $Q = S^2$. Moreover S^{-1} exists and is continuous. From Proposition 2 we get

$$U^* = S^2 U^{-1} S^{-2}$$

and after some calculations $(SUS^{-1})^* = (SUS^{-1})^{-1}$, that is SUS^{-1} is unitary and hence $\| U(t) \| \leqq \| S \| \cdot \| S^{-1} \|$ for all $t \geqq 0$.

The uniqueness of the solution of (2) implies that

$$U(n + t) = U(t) U(1)^n \qquad \text{for } n = 1, 2, \cdots$$

Hence (1) is stable if and only if there exists a constant M such that

$$\| U(1)^n \| \leqq M \qquad \text{for } n = 1, 2, \cdots$$

Since $\| U(1)^n \| \geqq (\nu(U(1)))^n$, where ν is the spectral radius, it follows that $\sigma(U(1)) \subset \{\lambda;\ |\lambda| \leqq 1\}$ is necessary for the stability of (1). When (S) is satisfied $\sigma(U(1))$is symmetric about the unit circle and hence $\sigma(U(1)) \subset \{\lambda;\ |\lambda| = 1\}$ is necessary.

Now we study the stability of (1) with a perturbation method, due to G. Borg [3] in the finite dimensional case. In order to state the next theorem we introduce some notations. Let the equation be

$$(3) \qquad u'(t) = (A_0(t) + B(t)) u(t)$$

We assume that

(a) $A_0(t)$ and $B(t)$ are periodic with period one.
(b) $A_0(t)$ is compact and antihermitian $(A_0(t)^* = - A_0(t))$ for all t.

Let further $U_0(t)$ be the unique solution of $U_0'(t) = A_0(t) U_0(t)$, $U_0(0) = I$. Suppose that

(c) $U_0(1)$ has only simple eigenvalues, λ_n, all $\neq 1$.
(d) $A_0(t) + B(t)$ satisfies condition (S).

Let further e_n be the eigenvector with norm one of $U_0(1)$ corresponding to the eigenvalue λ_n. Put

$$b_n^2 = \int_0^1 \| B(t) U_0(t) e_n \|^2 \, dt$$

$$K = \int_0^1 \exp\left[2 \int_t^1 \Phi(B(s)) ds\right] dt$$

$$r_n = 2^{-1} \inf_{k \neq n} |\lambda_n - \lambda_k| \,.$$

Theorem. *If* (a), (b), (c), (d) *and*

$$(e) \qquad K \cdot \sup_k \sum_{n=1}^{\infty} b_n^2 (|\lambda_k - \lambda_n| - r_k)^{-2} < 1$$

and

$$\text{(f)} \qquad \sum_{n=1}^{\infty} b_n^2 r_n^{-2} < \infty$$

are satisfied, then (3) *is stable.*

REMARK 1. The theorem is true if K and b_n are replaced by

$$K' = \exp\left\{2 \max_{0 \leq t \leq 1} \int_t^1 \Phi(B(s))ds\right\}, \qquad b'_n = \int_0^1 \| B(t) U_0(t) e_n \| \, dt .$$

It is easily seen that $K \leqq K'$ but $b'_n \leqq b_n$.

REMARK 2. If X is finite dimensional, then condition (f) is automatically fulfilled.

REMARK 3. $K \cdot \sum_1^\infty b_n^2 r_n^{-2} < 1$ implies both (e) and (f).

Proof of the theorem. The rather lengthy proof is divided in eight parts.

(i) *$U_0(t)$ is unitary for all t.*

A calculation shows that $U_0(t)^{-1} = V(t)^*$ where V is the unique solution of $V' = -A_0^*(t)V$, $V(0)=I$. But since $-A_0^* = A_0$ it follows that $U_0(t)^{-1} = U_0(t)^*$.

(ii) *$U_0(1) - I$ is compact.*

We have $U_0(1) - I = \int_0^1 A_0(t) U_0(t)dt$. The integral is compact because it is the limit of compact operators of the form $\sum_{i=1}^n A_0(t_i) U_0(t_i) \Delta t_i$.

From (i) and (ii) we conclude that $\{e_n\}_1^\infty$ is an orthonormal set and indeed a basis because $U_0(1) - I$ is compact and 1 is not an eigenvalue of $U_0(1)$. Further $\lim_{n\to\infty} \lambda_n = 1$. Since $U_0(t)$ is unitary

$$\| U_0(t) \| = \| U_0(t)^{-1} \| = 1 \quad \text{for all } t \text{ and } \quad | \lambda_n | = 1 .$$

Put $W(t) = U(t) - U_0(t)$. Further it is convenient to write $U(1) = U$, $U_0(1) = U_0$ and $W(1) = W$. Let C_k be the circumference of a circle with center λ_k and radius r_k.

(iii) $R_\lambda = (\lambda I - U)^{-1}$ *exists if* $\lambda \in \bigcup_1^\infty C_k$.

Put $R_\lambda^0 = (\lambda I - U_0)^{-1}$. For a λ such that R_λ^0 and $(I - WR_\lambda^0)^{-1}$

exist, we have

$$R_\lambda = R_\lambda^0(I - WR_\lambda^0)^{-1}$$

It is clear that R_λ^0 exists whenever $\lambda \in \bigcup_1^\infty C_k$ and if $\| WR_\lambda^0 \| < 1$ it follows that R_λ exists. Since $\{e_n\}_1^\infty$ is an orthonormal basis it follows that

$$\| WR_\lambda^0 \|^2 \leqq \sum_1^\infty \| WR_\lambda^0 e_n \|^2 .$$

But

$$\| WR_\lambda^0 e_n \| = | \lambda - \lambda_n |^{-1} \cdot \| We_n \|$$

since

$$R_\lambda^0 e_n = (\lambda - \lambda_n)^{-1} e_n .$$

One verifies that $W(t)$ satisfies the equation

$$W'(t) = (A_0(t) + B(t))\, W(t) + B(t)\, U_0(t)$$

which has the solution

$$W = W(1) = \int_0^1 U(1)\, U(s)^{-1} B(s)\, U_0(s) ds$$

Then we get

$$\| We_n \| \leqq \int_0^1 \| U(1)\, U(s)^{-1} \| \cdot \| B(s)\, U_0(s) e_n \| \, ds.$$

From Theorem 4 in [1] we find

$$\| U(1)\, U(s)^{-1} \| \leqq \exp \int_s^1 \Phi(A_0(t) + B(t)) dt .$$

But $\Phi(A_0(t) + B(t)) = \Phi(B(t))$ since $A_0(t)$ is antihermitian. We finally get

$$\| We_n \|^2 \leqq \left\{ \int_0^1 \exp \left[\int_s^1 \Phi(B(t)) dt \right] \| B(s)\, U_0(s) e_n \| \, ds \right\}^2$$
$$\leqq \int_0^1 \exp \left(2 \int_s^1 \Phi(B(t)) dt \right) ds \cdot \int_0^1 \| B(s)\, U_0(s) e_n \|^2 ds = K \cdot b_n^2 .$$

From condition (e) we conclude that

$$\sum_1^\infty \| WR_\lambda^0 e_n \|^2 \leqq K \cdot \sum_1^\infty b_n^2 \, | \lambda - \lambda_n |^{-2}$$
$$\leqq K \cdot \sup_k \sum_{n=1}^\infty b_n^2 (| \lambda_k - \lambda_n | - r_k)^{-2} < 1$$

and hence $\| WR_\lambda^0 \| < 1$ for all $\lambda \in \bigcup_1^\infty C_k$. Thus we have shown that R_λ exists if $\lambda \in \bigcup_1^\infty C_k$.

(iv) *$U - I$ is compact.*

From (iii) it follows that $\sum_1^\infty \| We_n \|^2 \leqq K \sum_1^\infty b_n^2 < \infty$ since (e) implies that $\sum_1^\infty b_n^2 < \infty$. Hence W belongs to the Schmidt class, cf. [5], and is compact. Further $U - I = (U_0 - I) + W$ is compact since $U_0 - I$ is compact (ii).

Put $D_n = \{\lambda;\ |\lambda - \lambda_n| < r_n\}$.

(v) *U has exactly one eigenvalue, α_n,* in *D_n and α_n is simple.*

Since $U - I$ is compact and $1 \notin D_n$ it follows that there is only a finite number of eigenvalues of U in D_n.

Now it is convenient to introduce a parameter μ in the equation. Thus we study $U' = (A_0(t) + \mu B(t))U$, $U(0) = I$ where $0 \leqq \mu \leqq 1$. A simple calculation shows that $R_\lambda(\mu)$ is a continuous function of μ. Hence the projection

$$E_n(\mu) = (2\pi i)^{-1} \int_{\sigma_n} R_\lambda(\mu) d\lambda$$

is also continuous in $[0, 1]$. Further we can find a partition

$$0 = \mu_1 < \mu_2 < \cdots < \mu_k = 1$$

such that

$$\| E_n(\mu_{\nu+1}) - E_n(\mu_\nu) \| < (2M)^{-1} \qquad \text{for } \nu = 1, 2, \cdots, k\ ,$$

where $M = \max_{0 \leqq \mu \leqq 1} \| E_n(\mu) \|$. According to a well known lemma (see [6, p. 424]) it follows that $\dim E_n(\mu_{\nu+1})X = \dim E_n(\mu_\nu)X$ if both sides are finite. This is the case here because $U(\mu) - I$ is compact for $0 \leqq \mu \leqq 1$ and D_n contains only a finite number of eigenvalues. Now $\dim E_n(0)X = 1$ and hence, $\dim E_n(1)X = 1$ by induction. Thus there is exactly one point $\alpha_n \in \sigma(U)$ in D_n and this α_n must be simple.

(vi) $|\alpha_n| = 1$.

Assume that $|\alpha_n| > 1$. Then it follows that $\bar{\alpha}_n^{-1} \in D_n$. But due to (S) we find that $\bar{\alpha}_n^{-1} \in \sigma(U)$ and there will be two points belonging to $\sigma(U)$ in D_n. This is impossible.

Assume now that $|\alpha_n| < 1$. If $\bar{\alpha}_n^{-1} \in D_u$ we can apply the same argument as above. If $\bar{\alpha}_n^{-1} \notin D_n$ it is easily seen that $\bar{\alpha}_n^{-1} \notin \sigma(U)$. In fact we show that if $\lambda \notin \bigcup_1^\infty D_k$ and $\lambda \neq 1$ it follows that $\lambda \notin \sigma(U)$. We need only consider λ with $|\lambda| > 1$. Let D_k be the circle closest to λ. Then it is clear that $\{\lambda - \lambda_n| \geqq || \lambda_n - \lambda_k| - r_k|$ for all n and we get

$$K \sum_1^\infty || WR_\lambda^0 e_n ||^2 \leqq K \sum_1^\infty b_n^2 | \lambda - \lambda_n |^{-2} \leqq K \sum_{n=1}^\infty b_n^2 (|\lambda_n - \lambda_k| - r_k)^{-2} < 1$$

due to (e). Hence R_λ exists.

Now we have proved that $\sigma(U)$ consists of simple eigenvalues on the unit circle with limit point 1. In the finite dimensional case it follows immediately that (3) is stable (see Boman [2]). In the infinite dimensional case we have to use condition (f).

Put $E_n(0) = E_n$ and $E_n(1) = F_n$. If $F_n e_n \neq 0$ we put $\varphi_n = F_n e_n$ and if $F_n e_n = 0$ we choose φ_n as an arbitrary eigenvector of U corresponding to α_n. We have $E_n e_n = e_n$ and $U\varphi_n = \alpha_n \varphi_n$.

(vii) $\sum_1^\infty || \varphi_n - e_n ||^2 < \infty$,

$$(F_n - E_n)e_n = (2\pi i)^{-1} \int_{c_n} (R_\lambda - R_\lambda^0) e_n d\lambda \ .$$

A calculation shows that

$$R_\lambda - R_\lambda^0 = R_\lambda^0 (I - WR_\lambda^0)^{-1} WR_\lambda^0 \ .$$

Thus

$$\begin{aligned} || (F_n - E_n)e_n || &\leqq (2\pi)^{-1} \int_{c_n} || R_\lambda^0 || \cdot || (I - WR_\lambda^0)^{-1} || \cdot || WR_\lambda^0 e_n || \cdot | d\lambda | \\ &\leqq (2\pi)^{-1} r_n^{-1} \sup_{\lambda \in c_n} (1 - || WR_\lambda^0 ||)^{-1} \cdot K^{1/2} b_n r_n^{-1} 2\pi r_n \\ &= \text{const} \cdot b_n r_n^{-1} \ . \end{aligned}$$

Here we used the fact that $|| R_\lambda^0 || = r_n^{-1}$ for all $\lambda \in c_n$. Then

$$\sum_1^\infty || (F_n - E_n)e_n ||^2 \leqq \text{const.} \sum_1^\infty b_n^2 r_n^{-2} < \infty \qquad \text{due to (f)} \ .$$

It follows that $F_n e_n = 0$ only for a finite number of n and hence

$$\sum_1^\infty || \varphi_n - e_n ||^2 < \infty \ .$$

We define a linear operator P by the relation $Px = \sum_1^\infty c_\nu \varphi_\nu$ where $x = \sum_1^\infty c_\nu e_\nu$ and $\sum_1^\infty | c_\nu |^2 < \infty$. We recall that an operator T is called injective if $Tx = 0$ implies $x = 0$.

(viii) *$I - P$ is compact and P is injective. Hence P^{-1} is continuous.*

$$\sum_1^\infty \| (I-P)e_n \|^2 = \sum_1^\infty \| e_n - \varphi_n \|^2 < \infty \qquad \text{due to (vii)} .$$

Thus $I - P$ belongs to the Schmidt class and is compact (see [5]). Assume now that $Px = \sum_1^\infty c_\nu \varphi_\nu = 0$. We apply the projection F_k and get

$$F_k \sum_1^\infty c_\nu \varphi_\nu = c_k F_k \varphi_k = c_k \varphi_k = 0$$

and $c_k = 0$ for every k. Hence $x = 0$ and P is injective.

Now we end the proof of the theorem. We have to estimate $\| U^n x \|$ for an arbitrary $x \in X$. Put $y = P^{-1}x$ and assume that $y = \sum_1^\infty a_\nu e_\nu$. We get $x = Py = \sum_1^\infty a_\nu \varphi_\nu$ and

$$U^n x = U^n P y = \sum_1^\infty a_\nu U^n \varphi_\nu = \sum_1^\infty a_\nu \alpha_\nu^n \varphi_\nu = P \sum_1^\infty a_\nu \alpha_\nu^n e_\nu .$$

Further

$$\begin{aligned} \| U^n x \| &\leqq \| P \| \cdot \{\sum_1^\infty | a_\nu \alpha_\nu^n |^2\}^{1/2} = \| P \| \cdot \{\sum_1^\infty | a_\nu |^2\}^{1/2} \\ &= \| P \| \cdot \| y \| \leqq \| P \| \cdot \| P^{-1} \| \cdot \| x \| , \end{aligned}$$

which implies that $\| U^n \| \leqq \| P \| \, \| P^{-1} \|$ for every n and the proof is finished.

REMARK 4. If $C = (K \cdot \sum_1^\infty b_n^2 r_n^{-2})^{1/2} < 2^{-1}$, then $\| U^n \| < (1 - 2C)^{-1}$.

Proof. From the proof of (iii) it follows that $\| W R_\lambda^0 \| \leqq C$ for all $\lambda \in \bigcup_1^\infty C_k$. Further we get

$$\| (F_n - E_n) e_n \| \leqq (1 - C)^{-1} K^{1/2} b_n r_n^{-1} < 1$$

for all n since

$$(1 - C)^{-2} K \sum_1^\infty b_n^2 r_n^{-2} = C^2 (1 - C)^{-2} < 1 .$$

Hence $F_n e_n \neq 0$ and $\varphi_n = F_n e_n$ for all n. Then

$$\| I - P \|^2 \leqq \sum_1^\infty \| \varphi_\nu - e_\nu \|^2 \leqq C^2 (1 - C)^{-2}$$

and

$$\| P \| \leqq 1 + C(1 - C)^{-1} = (1 - C)^{-1} .$$

Further

$$\| P^{-1} \| = \| (I - (I - P))^{-1} \| \leqq (1 - \| I - P \|)^{-1} \leqq (1 - C)(1-2C)^{-1} .$$

Finally

$$\| U^n \| \leqq \| P \| \cdot \| P^{-1} \| \leqq (1 - 2C)^{-1} .$$

An interesting application of the theorem is the second order equation

$$y'' + C(t)y = 0$$

in a Hilbert space Y, where $C(t)$ is selfadjoint. Put $X = Y \oplus Y$ and $u = \begin{pmatrix} y \\ y' \end{pmatrix}$. Then we get

$$u' = \begin{pmatrix} 0 & I \\ -C(t) & 0 \end{pmatrix} u .$$

This equation satisfies the symmetry condition (S) with $Q - \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

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THE ROYAL INSTITUTE OF TECHNOLOGY, STOCKHOLM.
AND
UNIVERSITY OF CALIFORNIA, BERKELEY