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SETS OF CONSTANT WIDTH

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A lower bound, better than those previously known, is given for the volume of a 3-dimensional body of constant width 1. Bounds are also given in the case of n-dimensional bodies of constant width 1, $n \geqq 4$. Short proofs of the known sharp bounds for such bodies in the Euclidean and Minkowskian planes are given using properties of mixed areas. An application is made to a measure of outer symmetry of sets of constant width in 2 and 3 dimensions.

Let K be a convex body in n-dimensional Euclidean space E_n. For each point u on the unit sphere S centered at the origin, let $b(u)$ be the distance between the two parallel supporting hyperplanes of K orthogonal to the direction. The function $b(u)$ is the "width function" of K. If $b(u)$ is constant on S, then we say K is a body of constant width.

If K_1 and K_2 are convex bodies, then $K_1 + K_2$ is the "Minkowski sum" or "vector sum" of K_1 and K_2 [5, p. 79]. The following useful theorem is well-known.

THEOREM 1. *A convex body K has constant width b if and only if $K + (-K)$ is a spherical ball of radius b.*

In the case of E_2, a number of special properties of sets of constant width are known-for example, the following theorem of Pàl (see [5, p. 127]).

THEOREM 2. *Any plane convex body B of constant width admits a circumscribed regular hexagon H.*

We shall be concerned with the following type of result, due to Blaschke and Lebesgue (see [1], [3], [4], [5, p. 128], [9]).

THEOREM 3. *Any plane convex body B of constant width* 1 *has area not less than $(\pi - \sqrt{3})/2$, the area of a Reuleaux triangle of width* 1.

The following short proof of Theorem 3 will set the stage for some later arguments.

Proof of Theorem 3. Let $A(K)$ denote the area of K. The "mixed area" of the plane convex bodies K_1 and K_2, $A(K_1, K_2)$, can be defined by the fundamental relation [5, p. 48],

$$A(K_1 + K_2) = A(K_1) + 2A(K_1, K_2) + A(K_2) \; . \tag{1}$$

The mixed area is monotonic in each argument [5, p. 86]. That is, if $K_1 \subset K_2$, then

$$A(K_1, K) \leqq A(K_2, K) \; . \tag{2}$$

It follows from (1), setting $K_1 = K_2 = K$, that

$$A(K, K) = A(K) \; . \tag{3}$$

Now let H be a regular hexagon circumscribed about B (Theorem 2). Assume the center of H is the origin, so $H = -H$. Then, using (2) and (3), we obtain

$$A(B, -B) \leqq A(H, -H) = A(H, H) = A(H) \; . \tag{4}$$

Thus, by (4), (1), and Theorem 1, we have

$$\begin{aligned} \pi &= A(B + (-B)) = 2A(B) + 2A(B, -B) \\ &\leqq 2A(B) + 2A(H) = 2A(B) + \sqrt{3} \; , \end{aligned} \tag{5}$$

from which the theorem follows.

It has long been conjectured that in E_3 any convex body of constant width 1 has volume at least that of a certain "tetrahedron of constant width" T (see [12, p. 81] for the construction of T). A computation of the volume of T leads to the conjecture,

Conjecture 1. Any 3-dimensional convex body of constant width 1 has volume not less than

$$\frac{2\pi}{3} - \frac{\pi\sqrt{3}}{4} \cos^{-1}(1/3) \approx .42 \; .$$

In §2 we shall prove that if B_3 is a 3-dimensional body of constant width 1, with volume $V(B_3)$, then

$$V(B_3) \geqq \beta = \frac{\pi}{3}(3\sqrt{6} - 7) \approx .365 \; . \tag{6}$$

Our proof of (6) will depend upon the following theorem of Blaschke [2].

THEOREM 4. *If a 3-dimensional convex body of constant width b has volume V and surface area S, then*

$$2V = bS - \frac{2\pi}{3} b^3 \; . \tag{7}$$

It follows from (7) that Conjecture 1 is equivalent to:

Conjecture 1′. Any 3-dimensional convex body of constant width 1 has surface area not less than

$$2\pi - \frac{\pi\sqrt{3}}{2}\cos^{-1}(1/3)\ .$$

Conjecture 1 can be transformed into still another form using the concept of "mixed surface area." Let $S(K)$ denote the surface area of K. If K_1 and K_2 are 3-dimensional convex bodies, then the surface area of $K_1 + K_2$ can be written in the form

$$S(K_1 + K_2) = S(K_1) + 2S(K_1, K_2) + S(K_2)\ ,$$

where $S(K_1, K_2)$ is the mixed surface area. Thus, if K has constant width 1, $4\pi = S(K + (-K)) = 2S(K) + 2S(K, -K)$. Hence Conjectures 1 and 1′ are equivalent to:

Conjecture 1″. Any 3-dimensional convex body of constant width 1 has mixed surface area not greater than

$$\frac{\pi\sqrt{3}}{2}\cos^{-1}(1/3)\ .$$

Firey [6] has proved that the volume V of an n-dimensional convex body of constant width 1 satisfies

$$V \geqq \frac{\pi - \sqrt{3}}{n!}\ ,\quad n \geqq 2\ . \tag{8}$$

In §2 we give the generally better lower bound,

$$V \geqq \lambda\omega_n \prod_{k=3}^{n}\left(1 - \sqrt{\frac{k}{2k+2}}\right),\quad n \geqq 3\ , \tag{9}$$

where ω_n is the volume of the unit ball in E_n, and

$$\lambda = \frac{\pi - \sqrt{3}}{2\pi}\ .$$

Let C be a centrally symmetric convex body centered at the origin in E_n. Then C is the unit sphere for a Minkowskian geometry. We say that a body K has "constant width relative to C" if $K + (-K)$ is homothetic to C. In particular, one says that K and C are "equivalent in width" in case $K + (-K) = 2C$, since the condition implies that K and C have the same width function. When C is the ordinary unit sphere we obtain the ordinary sets of constant width. Results about plane sets of relative constant width analogous to Theorem 2 and 3 are known (see [8], [10], and [11]). In § 3 we give a proof of the analogue of Theorem 3 in the Minkowski plane, using the same method as in our proof of Theorem 3.

Section 4 is devoted to some results on measures of outer symmetry for sets of constant width.

2. *Proof of* (6). Let B_3 be a 3-dimensional convex body of constant width 1. Then the inscribed sphere of B_3 has radius $\geqq 1 - \sqrt{3/8}$ (see [5, p. 125]). Assume that the center of the inscribed sphere is the origin. If $p(u)$ is the supporting function of B_3, then we have $p(u) \geqq 1 - \sqrt{3/8}$. Hence,

$$3V(B_3) = \int_{B_3} p(u)dS(u) \geqq (1 - \sqrt{3/8})S(B_3)\ , \tag{10}$$

where $S(B_3)$ is the surface area of B_3. Using Theorem 4 in (10), we obtain

$$3V(B_3) \geqq (1 - \sqrt{3/8})\left(2V(B_3) + \frac{2\pi}{3}\right)\ , \tag{11}$$

and (6) follows upon solving (11) for $V(B_3)$. This completes the proof.

Proof of (9). Define

$$\lambda_n = \inf V(K)\ , \tag{12}$$

as K ranges over all bodies of constant width 1 in E_n, and $V(K)$ is the volume of K. The Blaschke selection principle implies that there exist bodies of constant width 1 having volume λ_n. Let B be such a body, and let $p(u)$ be the support function of B with the center of its inscribed sphere as origin. Then, by [5, p. 125],

$$p(u) \geqq 1 - \sqrt{\frac{n}{2n+2}}\ .$$

Denoting the area element of B by $dS(u)$, we have,

$$n\lambda_n = nV(B) = \int_B p(u)dS(u) \geqq \left(1 - \sqrt{\frac{n}{2n+2}}\right)S(B)\ , \tag{13}$$

where $S(B)$ is the surface area of B. If we denote by B_u the projection of B onto a hyperplane orthogonal to u, then (see [5, p. 89])

$$S(B) = \frac{1}{\omega_{n-1}} \int V(B_u)du\ , \tag{14}$$

where $V(B_u)$ is the $(n-1)$-dimensional volume of B_u and the integration is over the surface of the unit sphere in E_n. Since B_u is an $(n-1)$-dimensional body of constant width 1, we have by (12) that $V(B_u) \geqq \lambda_{n-1}$. Hence

$$S(B) \geqq \frac{n\omega_n \lambda_{n-1}}{\omega_{n-1}} . \tag{15}$$

Combined with (13), this yields

$$\lambda_n \geqq \left(1 - \sqrt{\frac{n}{2n+2}}\right) \frac{\omega_n}{\omega_{n-1}} \lambda_{n-1} , \tag{16}$$

from which (9) follows. This completes the proof.

3. In this section, C is a centrally symmetric plane convex body centered at the origin 0. C admits an inscribed affine regular hexagon H (i.e., the affine image of a regular hexagon) having a side parallel to any specified direction [10]. Let the vertices of H be labelled $P_1, P_2, \cdots, P_6$ on the boundary of C traversed in the positive direction. A "relative Reuleaux triangle" is obtained by attaching arcs P_1P_2, P_3P_4, and P_5P_6 of the boundary of C to the respective sides P_1P_2, P_20, $0P_1$ of the triangle $0P_1P_2$. With H as above, a centrally symmetric hexagon circumscribed about C and touching C at P_i, $1 \leqq i \leqq 6$, is called a "C-hexagon." In fact, any hexagon homothetic to such a hexagon will be called a C-hexagon. Note that it case C is a circle, any C-hexagon is just a regular hexagon. One then sees that the following theorem from [10] is a Minkowskian geometry analogue of Theorem 2.

THEOREM 2′. *Let K be equivalent in width to C. Then K admits a circumscribed C-hexagon.*

Let H be a C-hexagon circumscribed about C. Let H' be the corresponding affine regular hexagon inscribed in C with its vertices on H. Then we shall show that

$$A(H) \leqq 4/3\, A(H') . \tag{17}$$

This follows from the following general lemma.

LEMMA 1. *Let H' be an affine regular hexagon inscribed in a centrally symmetric plane convex body K. Then*

$$A(K) \leqq 4/3\, A(H') . \tag{18}$$

Proof. By considering the support lines of K through the vertices of H', one sees that it suffices to prove (18) for K a centrally

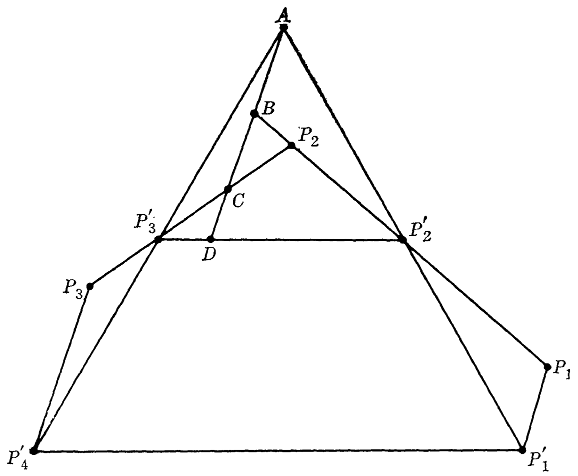

Figure 1.

symmetric hexagon H. Since the problem is affine invariant, one may even assume H' is a regular hexagon, although this does not really simplify matters. In Figure 1, P_1', P_2', P_3', P_4' are consecutive vertices of H', and P_1, P_2, P_3 are vertices of H. AD is drawn parallel to $P_4'P_3$, which is parallel to $P_1'P_1$ (the degenerate cases, where $P_3 = P_4'$ or $P_1 = P_1'$ are easily disposed of and will not be dwelt upon here). B is the intersection of P_1P_2 with AD, and C is the intersection of P_3P_2 with AD. Triangle $P_4'P_3P_3'$ is congruent to $P_3'CA$, and $P_1'P_1P_2'$ is congruent to $P_2'BA$. Hence the area of the pentagon $P_1'P_1P_2P_3P_4'$ is not greater than the area of triangle $P_1'AP_4'$, so the area of H is not greater than twice the area of $P_1'AP_4'$, which is precisely $4/3\,A(H')$. This completes the proof.

THEOREM 3'. *Any plane convex body K which is equivalent in width to C has area not less than that of some relative Reuleaux triangle equivalent in width to C.*

Proof. It is easy to check that the area of any relative Reuleaux triangle T equivalent in width to C is given by

$$A(T) = 2A(C) - 4/3\,A(H)\,, \tag{19}$$

where H is the affine regular hexagon inscribed in C on which the construction of T is based. Let H' be a C-hexagon circumscribed about K (Theorem 2'), let H'' be the translate of H' circumscribed about C, and let H be the corresponding affine regular hexagon inscribed in C with its vertices on H''. Let the center of H' be at the origin (which can be assumed by translating K) so $H' = -H'$. Then, proceeding as in the proof of Theorem 3, and using (17), we have

$$\begin{aligned} 4A(C) &= A(K + (-K)) = 2A(K) + 2A(K, -K) \\ &\leqq 2A(K) + 2A(H', -H') = 2A(K) + 2A(H') \\ &= 2A(K) + 2A(H'') \leqq 2A(K) + 8/3\, A(H)\ . \end{aligned} \tag{20}$$

Hence,

$$A(K) \geqq 2A(C) - 4/3\, A(H) = A(T)\ . \tag{21}$$

This completes the proof.

To prove that a relative Reuleaux triangle is the only body equivalent in width to C with minimum area requires a little more argument. A sketch of the proof is as follows. If K is such a body of minimum area, then equality must hold throughout (20). This means that $A(K, -K) = A(H')$ for a C-hexagon H' circumscribed about K. It follows that $A(-K, K) = A(H', K)$. If we let $p_1(\theta), p_2(\theta)$ be the support functions of K and H' respectively, with origin at the center of H', and let s_1 denote arclength along K, the last equation implies that

$$\int p_1(\theta + \pi) ds_1 = \int p_1(\theta) ds_1\ . \tag{22}$$

Equation (22) implies that K must pass through 3 alternate vertices of H', from which readily follows the fact that K is a relative Reuleaux triangle.

4. For any n-dimensional convex body K we define a "coefficient of outer symmetry," $\mu(K)$, as follows. Let S be a centrally symmetric convex body of minimum volume containing K. Then

$$\mu(K) = \frac{V(K)}{V(S)}\ , \tag{23}$$

Thus $\mu(K) \leqq 1$, and $\mu(K) = 1$ if and only if K is centrally symmetric. Sharp lower bounds for $\mu(K)$ are not known for $n \geqq 3$; however, it is known that $\mu(K) \geqq 1/2$ if K is 2-dimensional, with equality holding if and only if K is a triangle. A standing conjecture is that in E_n, $n \geqq 3$, $\mu(K) \geqq \mu(T)$, where T is a simplex.

THEOREM 5. *Let B be a plane convex body of constant width* 1. *Then $\mu(B) \geqq \mu(R)$, where R is a Reuleaux triangle, and equality holds only if B is a Reuleaux triangle.*

Proof. Let H be a regular hexagon circumscribed about B.

Then, using Theorem 3, we have

$$\mu(B) \geqq \frac{A(B)}{A(H)} \geqq \frac{A(R)}{A(H)} = \frac{\pi - \sqrt{3}}{\sqrt{3}} = .81 \cdots . \tag{24}$$

where R is a Reuleaux triangle of width 1. On the other hand, any centrally symmetric convex set S containing R must also contain an equilateral triangle T of side 1 and thus has area $\geqq 2A(T) = A(H)$. Hence

$$\frac{A(R)}{A(H)} = \mu(R) . \tag{25}$$

Equality can hold in (24) only if $A(B) = A(R)$, which happens only if B is a Reuleaux triangle (see end of § 3). This completes the proof.

It is known that any set K of constant width in E_3 admits a regular circumscribed octahedron J (see [7]). Suppose K has constant width 1, and let S be a centrally symmetric set of minimum volume containing K. Then, using (6),

$$\mu(K) = \frac{V(K)}{V(S)} \geqq \frac{\beta}{V(J)} = \frac{2\beta}{\sqrt{3}} \approx .42 . \tag{26}$$

Clearly one can obtain crude lower bounds, in this same fashion, in terms of λ_n and the volume of some centrally symmetric "covering body" J_n (one could, for example, use for J_n a sphere of radius $\sqrt{n/(2n+2)}$).

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