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AN INEQUALITY FOR THE DENSITY OF THE SUM OF SETS OF VECTORS IN n-DIMENSIONAL SPACE

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A Schnirelmann type density is defined for sets of "nonnegative" lattice points. If A, B and $C = A + B$ are such sets with densities α, β and γ respectively, then it is shown that $\gamma \geqq \beta/(1-\alpha)$ provided $\alpha + \beta < 1$.

1. Let n be a positive integer and let Q be the set of all vectors $r = (\rho_1, \cdots, \rho_n)$ where each ρ_i is a nonnegative integer and at least one ρ_i is positive. We define a partial order relation $<$ on Q where $r < s$ if and only if $\rho_i \leqq \sigma_i$ $(i = 1, 2, \cdots, n)$ with strict inequality holding for at least one index. Denote by $L(r)$ the set of all x in Q for which either $x < r$ or $x = r$.

A nonempty finite subset F of Q is called fundamental if, whenever $r \in F$, then $L(r) \subseteqq F$. For $A, X \subseteqq Q$ with X finite, let $A(X)$ denote the number of vectors in $A \cap X$. Then the (Kvarda) density of A is

$$\alpha = \text{glb} \frac{A(F)}{Q(F)}$$

where F ranges over all fundamental subsets of Q.

Let $B \subseteqq Q$ and define $A + B = \{a, b, a + b \mid a \in A, b \in B\}$ where addition of vectors is done coordinatewise. Let β and γ be the densities of B and $C = A + B$ respectively. Kvarda [1] has proved the inequaliy $\gamma = \alpha + \beta - \alpha\beta$ which for $n = 1$ was first proved by Landau and Schnirelmann. In this paper we prove $\gamma \geqq \beta/(1-\alpha)$ provided $\alpha + \beta < 1$. For $n = 1$, this has been proved by Schur [2].

2. Main results.

LEMMA 1. *Let $\bar{C}$ denote the complement of C in Q and suppose $\bar{C} \neq \Phi$. For a fundamental set F let F^* denote the set of maximal vectors of F with respect to the partial ordering $<$. Then*

$$\gamma = \text{glb} \frac{C(F)}{Q(F)}$$

where F ranges over all fundamental sets with $F^* \subseteqq \bar{C}$.

Proof. Let γ' denote this glb. Clearly $\gamma \leqq \gamma'$. Let G be any fundamental set. If $C(G) = Q(G)$ then $C(G)/Q(G) = 1 > \gamma'$. If $C(G) < Q(G)$ then $\bar{C} \cap G \neq \Phi$. In this case let F be the union of all sets $L(g)$ where $g \in \bar{C} \cap G$. Evidently F is a fundamental set, $F \subseteqq G$, and $F^* \subseteqq \bar{C}$. Thus,

$$\frac{C(G)}{Q(G)} = \frac{C(F) + C(G - F)}{Q(F) + Q(G - F)} = \frac{C(F) + Q(G - F)}{Q(F) + Q(G - F)} \geqq \frac{C(F)}{Q(F)} \geqq \gamma' ,$$

and so $\gamma \geqq \gamma'$.

LEMMA 2. *If F is a fundamental set with $F^* \subseteqq \bar{C}$, then $C(F) \geqq \alpha C(F) + B(F)$.*

Proof. Let $g_1, g_2, \cdots, g_k$ be the vectors of $\bar{C} \cap F$, indexed in such a way that

$$(1) \qquad g_i < g_j \quad \text{implies} \quad i < j .$$

Define $H_1 = L(g_1)$ and $H_{i+1} = L(g_{i+1}) - \bigcup_{j=1}^{i} H_j$. Then

(2) the H_i are disjoint,

(3) the union of the H_i is F, and

(4) for each i, $g_i \in H_i$.

Now (2) follows immediately by definition, and (3) from the fact that since $F^* \subseteqq \bar{C}$, we have for each $x \in F$, that $x \in L(g_i)$ for some i. To prove (4) notice that $g_i \notin H_i$ implies $g_i \in \bigcup_{j=1}^{i-1} H_j$, which in turn implies $g_i \in L(g_{j_0})$ for some $j_0 < i$, contrary to (1).

For each i let tH_i be the set of all vectors $g_i - x$ where x ranges over $H_i - \{g_i\}$. Then

(5) tH_i is either empty or is a fundamental set, and

(6) $Q(tH_i) = Q(H_i) - 1$.

To show (5) let z be an arbitrary vector in tH_i and let $y \in L(z)$. We have $g_i - z \leqq g_i - y < g_i$. Thus $g_i - y \in L(g_i) - \{g_i\}$ and, since $g_i - z \in H_i$, we have $g_i - y \in H_i - \{g_i\}$. Hence $g_i - (g_i - y) = y \in tH_i$ and so $L(z) \subseteqq tH_i$. Equation (6) is immediate.

Now, for each $a \in A \cap tH_i$, there exists a unique $x \in H_i - \{g_i\}$ such that $a = g_i - x$. Thus $x \in \bar{B}$. Also, by (4), we have $g_i \in \bar{B} \cap H_i$ and so

$$\begin{aligned} \bar{B}(H_i) &\geqq A(tH_i) + 1 \\ &\geqq \alpha Q(tH_i) + 1 \qquad \text{(from (5) and the definition of } \alpha) \\ &= \alpha(Q(H_i) - 1) + 1 \qquad \text{(from (6))} . \end{aligned}$$

Summing over i, using (2) and (3), we obtain

$$\begin{aligned} \bar{B}(F) &\geqq \alpha(Q(F) - k) + k \\ &= \alpha C(F) + \bar{C}(F) \end{aligned}$$

that is,

$$C(F) \geqq \alpha C(F) + B(F) .$$

THEOREM. *If* $\alpha + \beta < 1$ *then* $\gamma \geqq \beta/(1 - \alpha)$.

Proof. Since $\beta < 1 - \alpha$ and $\alpha < 1$, then $\beta/(1 - \alpha) < 1$. Hence if $\gamma = 1$, the theorem follows. If $\gamma < 1$, then $\bar{C} \neq \Phi$ and for any fundamental set F with $F^* \subseteqq \bar{C}$ we have by Lemma 2

$$C(F) \geqq \alpha C(F) + B(F) .$$

Hence,

$$\frac{C(F)}{Q(F)} \geqq \alpha \frac{C(F)}{Q(F)} + \frac{B(F)}{Q(F)} \geqq \alpha\gamma + \beta .$$

By Lemma 1 $\gamma \geqq \alpha\gamma + \beta$ that is, $\gamma \geqq \beta/(1 - \alpha)$.

3. **Remark.** A result of Kvarda [1] states that if $\alpha + \beta \geqq 1$ then $\gamma = 1$. This result and the above theorem can be used to prove quickly that if $\alpha > 0$ then A is a basis for Q, that is, $nA = Q$ for some n, where $iA = (i - 1)A + A$ for $i \geqq 2$. Thus let γ_i denote the density of iA and assume that $nA \neq Q$ for all n. Then, for all k, $\gamma_k + \alpha < 1$, and so

$$\gamma_{k+1} \geqq \frac{\gamma_k}{1 - \alpha} \geqq \frac{\gamma_{k-1}}{(1 - \alpha)^2} \geqq \cdots \geqq \frac{\gamma_1}{(1 - \alpha)^k} = \frac{\alpha}{(1 - \alpha)^k}$$

But, for k sufficiently large, $(\alpha/(1 - \alpha)^k) \geqq 1$, a contradiction.

REFERENCES

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