PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 2, 1966

A THEOREM ON ONE-TO-ONE MAPPINGS

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Let X be a locally connected generalized continuum with the property that the complement of each compact set has only one nonconditionally compact component. The author proves the following theorem. If f is a one-to-one mapping of X onto Euclidean 2-space, then f is a homeomorphism.

An example of K. Whyburn implies that if f is a one-to-one mapping of X onto Euclidean n-space ($n \geqq 3$), then X can have many nice properties any yet f need not be a homeomorphism. However the complement of a compact set in the domain space of his example may have more than one nonconditionally compact component.

It is interesting to note that a characterization of closed 2-cells in the plane is obtained in the course of proving the theorem.

Positive results in connection with the following problem would be useful in classifying mappings from a Euclidean space into itself. "What properties must a topological space X have before one can conclude that every one-to-one mapping f of X into a Euclidean space E^n of dimension n is a homeomorphism?" A very general theorem of this type was supposedly obtained in [2]. However, several counterexamples have been obtained which show the main theorems of [2] to be false. One of these is an example of K. Whyburn [6], which implies that if $n \geqq 3$, X may have many nice properties, yet f need not be a homeomorphism. We prove that if the Euclidean space has dimension two, the mapping f is onto, and X has appropriate properties, then f is indeed a homeomorphism. It is interesting to note that we assign a property to the space X which is not a property of the domain space of the example in [6].

2. Notation.

A mapping is a continuous function. A generalized continuum is a connected, locally compact, separable metric space. The cyclic element theory used is that of reference [4]. A set A in a topological space is conditionally compact if its closure is a compact set. A dendrite is a compact locally connected generalized continuum containing no simple closed curve. A topological line is a homeomorphic image of the real line. A topological ray is a homeomorphic image of a ray in the real line.

3. Theorem and proof.

THEOREM. *Let X be a locally connected generalized continuum* *with the property that the complement of each compact set has only one nonconditionally compact component. If $f(X) = E^2$ is a one-to-one mapping, then f is a homeomorphism.*

Proof. The proof consists of proving a series of five statements concerning the structure of X if f is not a homeomorphism. Then in (vi) with the aid of a theorem of G. T. Whyburn [5], a contradiction is obtained.

(i) X contains simple closed curves.

Proof of (i). The space X has the representation, $X = \bigcup_{i=1}^{\infty} A_i$, where each A_i is a locally connected continuum. If X has no simple closed curve, then no A_i contains a simple closed curve. Thus each A_i is a dendrite and therefore has dimension one. Using the sum theorem for dimension, we obtain dim $\bigcup_{i=1}^{\infty} f(A_i) \leqq 1$, but $\bigcup_{i=1}^{\infty} f(A_i) = E^2$ and dim $E^2 = 2$. Clearly then each such X must contain simple closed curves.

(ii) Every simple closed curve J in X separates X and is the boundary of an open two cell which is an open subset of X.

Proof of (ii). For a simple closed curve J in X, $f(J)$ is a simple closed curve. Since $f(J)$ separates E^2, its inverse image J separates X. The complement of J, X-J, can have at most countably many components, C_i, $i = 0, 1, 2, \cdots$, and only one of these, say C_0, is not conditionally compact. Each $f(C_i)$, $i \neq 0$, is closed in $E^2 - f(J)$ and each $f(C_i)$, $i = 0, 1, 2, \cdots$ is either in the bounded component W or the unbounded component M of $E^2 - f(J)$. The set $f(C_0)$ is not contained in W for this would imply that M is the countable union of pairwise disjoint bounded closed (in M) sets $f(C_{n_k})$, $k = 1, 2, \cdots$. No arcwise connected space has such a representation hence $f(C_0) \subset M$. Applying the same theorem to W shows there is one and only one C_i, $i \neq 0$, for which $f(C_i) \subset W$ and hence $f(C_i) = W$. It easily follows that $f(F_rC_i) = f(J)$ and therefore $F_rC_i = J$.

(iii) Each compact nondegenerate cyclic element of X is topologically a closed 2-cell.

Proof of (iii). Let C be a compact nondegenerate cyclic element of X and note by (ii) that every simple closed curve in C is the boundary of an open 2-cell of C. Since f/C is a homeomorphism we can assume that C is a subset of E^2.

Let H be the set of points of C that are interior to an open 2-cell of C. By cyclic connectedness of C, H is dense in C. To show H is connected let a and b be distinct points of H and let J_1 and J_2 be disjoint simple closed curves in C that bound nonintersecting open 2-cells C_1 and C_2 containing a and b respectively. By cyclic connectedness of C there exist mutually exclusive arcs 1_1 and 1_2 in C with $1_1 \cap (C_1 \cup J_1) = 1_1 \cap J_1 = x_{11}$, $1_1 \cap (C_2 \cup J_2) = 1_1 \cap J_{12} = x_{12}$, $1_2 \cap (C_1 \cup J_1) = 1_2 \cap J_2 = x_{21}$, and $1_2 \cap (C_2 \cup J_2) = 1_2 \cap J_2 = x_{22}$. The set $1_1 \cup (x_{11}x_{21}) \cup 1_2 \cup (x_{12}x_{22})$, where $(x_{11}x_{21})$, $(x_{12}x_{22})$ are arcs on J_1 and J_2 are respectively, is a simple closed curve in C. The proper choice of arcs $(x_{11}x_{21})$ and $(x_{12}x_{22})$ on J_1 and J_2, respectively, gives a simple closed curve J_0 in C that bounds an open 2-cell C_0 which contains both a and b.

We use the Zoretti Theorem, p. 109, [4], to prove C–H is connected. Suppose C–H is not connected and K is one of its compact components. By Zoretti's Theorem there is a simple closed curve J_3 in E^2 enclosing K and not enclosing C–H and is such that $J_3 \cap (C-H) = \varnothing$. The set $J_3 \cap C = J_3 \cap H$ is not empty and is both open and closed in J_3. Hence $J_3 \subset H$ and this implies $K \subset H$ which is false.

Let x and y be distinct points of C–H. By the cyclic connectedness of C and the connectedness of H there is a simple arc (xy) in C with $(xy) \cap (C-H) = x \cup y$. Suppose this arc does not separate C and let $z \in (xy)$, $z \neq x$, $z \neq y$. Since $z \in H$ there is a closed 2-cell C_4 in H with boundary J_4 such that z is interior to C_4 and (xy) separates C_4 into two connected sets A and B. Let $a \varepsilon A$ and $b \varepsilon B$ and suppose (ab) is a simple arc in C–(xy).

In C_4 determine an arc azb such that (ab) union azb is a simple closed curve J_5. The curve J_5 is the boundary of a closed 2-cell C_5 in C. The 2-cell C_5 contains points of A and B and hence points of one of the subarcs (xz) or (zy) of (xy) other than z. Since J_5 meets (xy) only in the point z, at least one of x or y is interior to C_5. This contrary to the choice of x and y. Therefore, each such arc spanning C–H in C separates C. Furthermore, H–(xy) has only two components and hence C–(xy) has only two components since H is dense in C. Also, each component of C–(xy) contains points of C–H, otherwise there would exist a bounded open subset of the plane with a simple arc as its frontier. Thus each pair of points x, y of C–H separates C–H and therefore C–H is a simple closed curve J. Clearly H is the open two cell of C bounded by J.

In order to make repeated use of a theorem in [5] we set up the following notation. Let $f(X) = Y$ be a one-to-one continuous mapping of one locally compact separable metric space onto another. Let S be the set of points in X at which f is a local homeomorphism and let T be its complement. From a result in [3] the set S is open, T is closed, $f(S)$ is an open dense set in Y. The sets S and T will be used in the remaining parts of the proof. The following is a theorem of G. T. Whyburn [5].

THEOREM A. *Let X be a locally compact arcwise connected separable metric space, let Y be a locally connected generalized continuum. If $f(X) = Y$ is a one-to-one continuous function which is not a homeomorphism, then there exists a topological ray R in X with $f(R)$ a simple closed curve in Y. Moreover, if r is the initial point of R, there is a subray R' of R such that $f(R' \cup r)$ is a simple arc and $R' \subset S$.*

(iv) There is only one noncompact cyclic element in X.

Proof of (iv). If there were two or more noncompact cyclic elements then one could find a compact set (namely a point) whose complement would necessarily have two or more nonconditionally compact components. This is contrary to part of the hypotheses on X.

If all the cyclic elements were compact then by (iii), all the true cyclic elements would be closed 2-cells. Thus S would be the union of open 2-cells. By Theorem A there is a ray R' and a point r not a R' such that $f(R' \cup r)$ is a simple arc and $R' \subset S$. Thus R' must be a closed subset of X which is entirely in an open 2-cell and this is not possible.

(v) Let M be the noncompact cyclic element of X and let $B = M \cap T$. The set B is a topological line.

Proof of (v). As in the proof of (iii), the set M–B is connected. For two distinct points a and b of B there is a simple arc $[ab]$ in M with $[ab] \cap B = a \cup b$. Using the techniques of the proof of (iii), it follows that the arc $[ab]$ separates M into two connected sets. The closure of the conditionally compact component D is cyclically connected and every simple closed curve in $\bar{D}$ bounds an open 2-cell of $\bar{D}$. Thus by (iii), $\bar{D}$ is a closed 2-cell and this implies that there is a simple arc (ab) which is entirely in B. Moreover, the set (ab)–$\{a \cup b\}$ is an open subset of B. If c is any other point of B not on (ab), then there is a simple arc joining c to a and the first point (ordered from c to a) in which it meets (ab) can only be a or b. Thus, either a or b is in an open one cell which is an open subset of B. It follows that every point of B with the possible exception of at most two points is in an open one cell which is an open subset of B. That is, B is a simple arc, a topological ray or a topological line. The set B cannot have a point d which is not interior to a one cell of B for this would imply that M is not locally compact at d.

(vi) Completion of proof.

The structure of the space X is now clear in the sense that certain properties can be assigned to the true cyclic elements. Also, each component of the complement of the noncompact cyclic element M is conditionally compact and has only a single point of B as its frontier.

There is a ray R_0 in X with initial point r_0 such that $f(R_0)$ is a simple closed curve bounding a closed 2-cell C_0. From the proof of Theorem A and the structure of X we can assume R_0 meets B in only one point x_0. The set $f^{-1}(C_0) = N_0$ is closed, connected, locally connected, and contains one of the two components of $M\text{-}R_0$. Let $y \in B \cap N_0$ and let (x_0y) be the simple arc in B. Let K be (x_0y) union the conditionally compact components of $X\text{-}(x_0y)$. The set K is compact and connected. The set $f(K \cap T)$ is compact in C_0 so that $(f(T) \cap C_0)\text{-}f(K \cap T)$ is an open subset of $f(T) \cap C_0$. Thus, in applying the proof of Theorem A to the map f/N_0: $N_0 \to C_0$ we can use the points of $(f(T) \cap C_0)\text{-}f(K \cap T)$ to get a ray R_1 with the property that $R_1 \subset N_0\text{-}K$. Assume the initial point of R_1 is r_1, $R_1 \cap B = x_1$, C_1 is the closed 2-cell bounded by $f(R_1)$, and $N_1 = f^{-1}(C_1)$. The set N_1 is connected, locally connected, and $N_1 \cap K = \varnothing$. In fact, the arc (x_0x_1) in B maps onto an arc in the closed annular region determined by $f(R_0)$ and $f(R_1)$. Also implied is that a sequence of rays $R_0, R_1, R_2, \cdots$ can be obtained such that $\limsup R_n \cap T = \varnothing$. We can also suppose the rays were chosen so that a monotone sequence of locally connected generalized continua, $N_0 \supset N_1 \supset N_2 \supset \cdots$ with corresponding closed 2-cell images $C_0 \supset C_1 \supset C_2 \supset \cdots$ is obtained. For each i, $i = 0, 1, 2, \cdots$, the set $C_i \cap f(T)$ is nonempty and compact. Thus, $L = \bigcap_{i=0}^{\infty} [C_i \cap f(T)]$ is not empty and for $y \in L$ there exists an $x \in (T \cap N_i)$, $i = 0, 1, 2, \cdots$ such that $f(x) = y$. However, by the construction of the N_i, $\bigcap_{i=0}^{\infty} (N_i \cap T) = \varnothing$.

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Received August 6, 1965. Presented to the American Mathematical Society, April 15, 1965.

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