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ON OPERATORS WHOSE SPECTRUM LIES ON A CIRCLE OR A LINE

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The purpose of this note is to prove the following theorem.

THEOREM. Let N be a bounded linear operator on Hilbert space H satisfying

$$(1) \qquad \| NT - TN \| = \| N^* T - TN^* \|$$

for all bounded linear operators T. Then N is (obviously) normal and the spectrum of N lies on a circle or straight line.

Here N^* denotes the adjoint of the operator N.

It is clear that if S is a unitary or self-adjoint operator and α and β are complex numbers, then $N = \alpha I + \beta S$ satisfies (1). The theorem asserts that the converse is also true.

After noting that in dimensions two and three the theorem is trivially true, we proceed to the first of two parts of the proof.

I. Dimension four. Let H be four dimensional Euclidean space. Since a normal operator N is unitarily equivalent to a diagonal matrix it is no restriction to assume that N has the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

where the λ_i's are the eigenvalues of N. Consider the matrix

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$NT - TN = \begin{pmatrix} 0 & 0 & (\lambda_1 - \lambda_3) & (\lambda_1 - \lambda_4) \\ 0 & 0 & (\lambda_2 - \lambda_3) & i(\lambda_2 - \lambda_4) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\| NT - TN \|^2$ is the largest eigenvalue of the matrix

$$A = \begin{pmatrix} |\lambda_1-\lambda_3|^2+|\lambda_1-\lambda_4|^2 & \begin{matrix}(\lambda_1-\lambda_3)\overline{(\lambda_2-\lambda_3)} \\ -i(\lambda_1-\lambda_4)\overline{(\lambda_2-\lambda_4)}\end{matrix} & 0 & 0 \\ \begin{matrix}\overline{(\lambda_1-\lambda_3)}(\lambda_2-\lambda_3) \\ +i\overline{(\lambda_1-\lambda_4)}(\lambda_2-\lambda_4)\end{matrix} & |\lambda_2-\lambda_3|^2+|\lambda_2-\lambda_4|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, $\|N^*T - TN^*\|^2$ is the largest eigenvalue of a matrix B which has the same form as A but with λ_i replaced by $\bar{\lambda}_i$, $i = 1, 2, 3, 4$.

Let $P(X)$ and $P^*(X)$ be the characteristic polynomials of A and B, respectively. Then from the form of the matrices A and B one sees that the polynomials $P(X)$ and $P^*(X)$ differ only in the coefficient of X^2. Therefore $\|NT - TN\| = \|N^*T - TN^*\|$ if, and only if, $P(X)$ and $P^*(X)$ are equal and a routine computation shows that this holds if, and only if,

$$\begin{aligned}&[\overline{(\lambda_1-\lambda_3)}(\lambda_2-\lambda_3)+i\overline{(\lambda_1-\lambda_4)}(\lambda_2-\lambda_4)] \\ &\qquad\times[(\lambda_1-\lambda_3)\overline{(\lambda_2-\lambda_3)}-i(\lambda_1-\lambda_4)\overline{(\lambda_2-\lambda_4)}] \\ &\quad=[(\lambda_1-\lambda_3)\overline{(\lambda_2-\lambda_3)}+i(\lambda_1-\lambda_4)\overline{(\lambda_2-\lambda_4)}] \\ &\qquad\times[\overline{(\lambda_1-\lambda_3)}(\lambda_2-\lambda_3)-i\overline{(\lambda_1-\lambda_4)}(\lambda_2-\lambda_4)]\end{aligned}$$

which reduces to the condition that $\overline{(\lambda_1-\lambda_4)}\overline{(\lambda_2-\lambda_3)}(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)$ be real. The latter holds if, and only if, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ lie on a line or circle, which proves the theorem in dimension four.

Let (a_1, a_2, a_3, a_4) be a vector in four dimensional H. Since

$$\begin{aligned}&(NT-TN)(a_1, a_2, a_3, a_4) \\ &\quad=((\lambda_1-\lambda_3)a_3+(\lambda_1-\lambda_4)a_4, (\lambda_2-\lambda_3)a_3+i(\lambda_2-\lambda_4)a_4, 0, 0)\,,\end{aligned}$$

we remark that we have just shown that

$$(2)\qquad \sup_{\Sigma|a_i|^2=1}[|(\lambda_1-\lambda_3)a_3+(\lambda_1-\lambda_4)a_4|^2+|(\lambda_2-\lambda_3)a_3+i(\lambda_2-\lambda_4)a_4|^2]$$

is the largest root of $P(X) = 0$, and also that

$$(3)\qquad \sup_{\Sigma|a_i|^2=1}[|\overline{(\lambda_1-\lambda_3)}a_3+\overline{(\lambda_1-\lambda_4)}a_4|^2+|\overline{(\lambda_2-\lambda_3)}a_3+i\overline{(\lambda_2-\lambda_4)}a_4|^2]$$

is the largest root of $P^*(X) = 0$.

II. Higher dimensions. Now let H be a finite or infinite dimensional Hilbert space of dimension > 4. Let N be a normal operator satisfying $\|NT - TN\| = \|N^*T - TN^*\|$ for all T. Suppose N has spectrum S and spectral representation $N = \int_S \lambda dE_\lambda$. Let

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be distinct points of S. (If S has no more than three points then the theorem is trivially true.) Choose an $\varepsilon > 0$ but smaller than the minimum of the distances $|\lambda_i - \lambda_j|$, $i \neq j$; let $S_k = S \cap \{\lambda \,||\, \lambda - \lambda_k| < \varepsilon\}$, $k = 1, 2, 3, 4$. Let $E_k = \int_{S_k} dE_\lambda$ and choose $y \in H$ with $\| E_k y \| = 1$ for $k = 1, 2, 3, 4$ and let $x_k = E_k y$. Then $(x_i, x_j) = 0$ if $i \neq j$ and each x_k is an approximate eigenvector in the sense that

$$\| Nx_k - \lambda_k x_k \| = \left\| \int_{S_k} \lambda dE_\lambda x_k - \lambda_k \int_{S_k} dE_\lambda x_k \right\| \\ \leqq \max_{\lambda \in S_k} |\lambda - \lambda_k| < \varepsilon .$$

We can therefore write $Nx_k = \lambda_k x_k + \varepsilon u_k$ where $\| u_k \| \leqq 1$. Similarly $N^* x_k = \bar{\lambda}_k x_k + \varepsilon u'_k$, $\| u'_k \| \leqq 1$.

Define an operator T on H by

$$Tx = [(E_3 x, x_3) + (E_4 x, x_4)]x_1 + [(E_3 x, x_3) + i(E_4 x, x_4)]x_2 .$$

Now for $x \in H$,

$$NTx = [(E_3 x, x_3) + (E_4 x, x_4)]\lambda_1 x_1 \\ + [(E_3 x, x_3) + i(E_4 x, x_4)]\lambda_2 x_2 + \delta_1(x) ,$$

where $\| \delta_1(x) \| \leqq 4\varepsilon \| x \|$. Also

$$T(Nx) = [(E_3 Nx, x_3) + (E_4 Nx, x_4)]x_1 \\ + [(E_3 Nx, x_3) + i(E_4 Nx, x_4)]x_2 .$$

Since $NE_k = E_k N$, $k = 1, 2, 3, 4$, it follows that

$$TNx = [(E_3 x, \bar{\lambda}_3 x_3) + (E_4 x, \bar{\lambda}_4 x_4)]x_1 \\ + [(E_3 x, \bar{\lambda}_3 x_3) + i(E_4 x, \bar{\lambda}_4 x_4)] + \delta_2(x) ,$$

where $\| \delta_2(x) \| \leqq 4\varepsilon \| x \|$.

Thus

$$\| (NT - TN)x \| \\ = \| [(\lambda_1 - \lambda_3)(E_3 x, x_3) + (\lambda_1 - \lambda_4)(E_4 x, x_4)]x_1 \\ + [(\lambda_2 - \lambda_3)(E_3 x, x_3) + i(\lambda_2 - \lambda_4)(E_4 x, x_4)]x_2 + \gamma_1(x) \| ,$$

where $\| \gamma_1(x) \| \leqq 8\varepsilon \| x \|$.

Similarly

$$\| (N^* T - TN^*)x \| \\ = \| [(\overline{\lambda_1 - \lambda_3})(E_3 x, x_3) + (\overline{\lambda_1 - \lambda_4})(E_4 x, x_4)]x_1 \\ + [(\overline{\lambda_2 - \lambda_3})(E_3 x, x_3) + i(\overline{\lambda_2 - \lambda_4})(E_4 x, x_4)]x_2 + \gamma_2(x) \| ,$$

where $\| \gamma_2(x) \| \leqq 8\varepsilon \| x \|$.

Since x_1 and x_2 are orthogonal,

$$\begin{aligned}
&\| NT - TN \|^2 \\
&\quad = \sup_{\|x\|=1} [| (\lambda_1 - \lambda_3)(E_3x, x_3) + (\lambda_1 - \lambda_4)(E_4x, x_4) |^2 \\
&\qquad + | (\lambda_2 - \lambda_3)(E_3x, x_3) + i(\lambda_2 - \lambda_4)(E_4x, x_4) |^2] + \delta
\end{aligned}$$

and

$$\begin{aligned}
&\| N^*T - TN^* \|^2 \\
&\quad = \sup_{\|x\|=1} [| (\overline{\lambda_1 - \lambda_3})(E_3x, x_3) + (\overline{\lambda_1 - \lambda_4})(E_4x, x_4) |^2 \\
&\qquad + | (\overline{\lambda_2 - \lambda_3})(E_3x, x_3) + i(\overline{\lambda_2 - \lambda_4})(E_4x, x_4) |^2] + \delta' ,
\end{aligned}$$

where $\delta, \delta' \leqq 8\varepsilon$.

Thus if $\| NT - TN \| = \| N^*T - TN^* \|$, we obtain by (2) and (3) that the largest zeros of $P(X)$ and $P^*(X)$ differ by at most 16ε. Therefore all the coefficients of $P(X)$ and $P^*(X)$ are close to each other and in particular the imaginary part of

$$(\overline{\lambda_1 - \lambda_4})(\overline{\lambda_2 - \lambda_3})(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)$$

is bounded by a function of ε which goes to zero with ε. Letting $\varepsilon \to 0$ we again find that $(\overline{\lambda_1 - \lambda_4})(\overline{\lambda_2 - \lambda_3})(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)$ must be real and so $\lambda_1, \lambda_2, \lambda_3$ and λ_4 lie on a line or a circle. Since the λ_i's were arbitrary, the proof is concluded.

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