

A GENERALIZATION OF THE WILDER ARCS

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Fox and Harrold first used the words "Wilder arc" to describe a wild arc in euclidean 3-space E^3 which is the union of two tame arcs meeting only in a common endpoint and which is locally peripherally unknotted (L.P.U.) at this point of intersection. Thus there are imposed (a) conditions upon the embeddings of subarcs of the wild arc and (b) conditions upon the manner in which the subarcs meet. The following definition gives only conditions of type (a): An arc $A \subset E^3$ is almost tame if each point of A lies on a tame subarc of A . Clearly, every Wilder arc is almost tame.

The chief result characterizes the set W of points on an almost tame arc at which the arc can fail to be locally tame. In particular, W is shown to be homeomorphic to a closed countable set W' on the unit interval with the property that a point $x \in W'$ either is the first or last point of W' or x has either an immediate predecessor or an immediate successor. Two further results discuss special cases.

LEMMA 0. *If the arc $A \subset E^3$ is almost tame, then A is locally tame at each of its endpoints and hence is L.P.U. at these points.*

This lemma follows immediately from the definition of an almost tame arc.

LEMMA 1. *Let C be an uncountable set in the closed unit interval $I = [0, 1]$ in E^1 . Then there is a point in C which is a limit point of C both from below and from above.*

This follows from Theorem 6, Chapter 1 of [4].

LEMMA 2. *Let the arc $A \subset E^3$ be almost tame and let W denote the set of points of A at which A is not locally tame. Then W is a closed countable subset of A . Furthermore, if $h: I \rightarrow A$ is a homeomorphism, then under the order relation induced on A by h , one has the property that, for each $x \in W$, (i) x is the first point of W or x is the last point of W or (ii) x has either a unique predecessor or a unique successor.*

Proof. The set of points at which A is locally tame is open so W is obviously closed.

Let x be a point of W . By Lemma 0, we have $0 < h^{-1}(x) < 1$. By assumption of almost tameness, x lies on a tame subarc of A .

Hence there is a maximal connected subset U of A that contains x but no other point of W and \bar{U} is a nondegenerate arc. If neither endpoint of \bar{U} is in W , then obviously $\bar{U} = A$ and x is the only point in W . Let y be the left-hand endpoint of \bar{U} , i.e. $h^{-1}(y) \leq h^{-1}(x)$. If $h^{-1}(y) < h^{-1}(x)$ and if $y \notin W$, then there is no point of W between $h(0)$ and x so x is the first point of W . If $h^{-1}(y) < h^{-1}(x)$ and $y \in W$, then clearly y is the immediate predecessor of x in W . If $y = x$, consider the right-hand endpoint z of \bar{U} . Since \bar{U} is nondegenerate $h^{-1}(x) < h^{-1}(z)$. If $z \notin W$, then there is no point of W between x and $h(1)$; whence x is the last point of W . If $z \in W$, then z is the immediate successor of x in W .

The set W can not have a limit point both from above and from below under the circumstances just established. Hence by Lemma 1, W is countable.

THEOREM 1. *Let $W \subset I \subset E^3$ be a closed countable set with the property that a point $x \in W$ either is the first or last point of W or x has either an immediate predecessor or an immediate successor. Then there is an almost tame arc $A \subset E^3$ such that W is precisely the set of points at which A is not locally tame.*

Proof. Let $B = \{a_i\} \subset W$ be the set of isolated points of W . In a small open sphere U_i containing a_i and no other points of W , we introduce a Wilder arc J_i [1] as follows: Let a_i be the point at which J_i is not locally tame and replace the interval $\bar{U}_i \cap I$ by J_i so that J_i has the same endpoints as does $\bar{U}_i \cap I$ and so that J_i lies in a symmetric cylinder of radius less than $1/i$ having $\bar{U}_i \cap I$ as axis of symmetry while $(I - U_i) \cup J_i$ is an arc. If this is done for each U_i the result is an arc A . Clearly the set B is contained in the set of points at which A is not locally tame. Furthermore each limit point of W is a limit point of the isolated points of W hence A fails to be locally tame at each point of W and only at these points. Clearly every point of W lies on a closed subarc of A which is either homeomorphic to J_i or is an interval. Hence A is almost tame.

THEOREM 2. *Let $A \subset E^3$ be an almost tame arc such that each subarc of A is also almost tame. Then A is a union of a finite number of tame arcs.*

Proof. Again let W be the set of points of A at which A is not locally tame and suppose that W has a limit point x . By Lemma 2, x is a limit point of points of W from only one side of x . Select a small subarc J in A with x as an endpoint and so that x is a limit point of $J \cap W$. The subarc J is then not almost tame. Hence W

can contain no limit points and so is finite. It follows immediately that A is a union of finitely many tame arcs, as asserted.

It should be noted that the construction of the arc A in Theorem 1 gives rise to an arc which is L.P.U. at each of its points. We may define a *generalized Wilder arc* to be an almost tame arc which is L.P.U. at each of its points. Clearly Theorems 1 and 2 then apply to such arcs. Indeed, we remark that a generalized Wilder arc is necessarily cellular. The converse is not true, of course. The most one can say in this direction is as follows:

THEOREM 3. *Suppose the almost tame arc $A \subset E^3$ is cellular. Then every subarc J of A which is locally tame at its interior points is also locally tame at one of its endpoints.*

Proof. From [3] it follows that each subarc of A is cellular. Given a cellular subarc J as in the theorem, [2] then asserts that J is locally tame at one endpoint.

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