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INVERSE LIMITS OF INDECOMPOSABLE CONTINUA

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Let $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ denote an inverse limit system of continua, with inverse limit space X_∞. Capel has shown that if each X_λ is an arc (simple closed curve), then X_∞ is an arc (simple closed curve) provided that Λ is countable and the bonding maps are monotone and onto. It is shown in this paper that a similar result holds when each X_λ is a pseudo-arc. In fact, the restrictions that the bonding maps be monotone and onto may be deleted.

Two theorems are proved which lead to this result. First, it is shown that if the maps of an inverse system of indecomposable continua are onto, then the limit space is an indecomposable continuum. Next, it is shown that with no restrictions on the bonding maps, a similar statement is true for hereditarily indecomposable continua.

1. Definitions and notation. All spaces are assumed to be Hausdorff. The notation $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ represents an inverse limit system with factor spaces X_λ, bonding maps $f_{\lambda\mu}$ and directed set Λ. The inverse limit space of the system $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ is denoted by X_∞. Definitions of these terms may be found in [2]. For each $\lambda \in \Lambda$, Π_λ denotes the projection function of $P_{\lambda\in\Lambda} X_\lambda$ onto X_λ, restricted to X_∞.

A *continuum* is a compact connected Hausdorff space. A continuum is *indecomposable* if it cannot be expressed as the union of two proper subcontinua. It is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

A *chain* is a finite collection of open sets $U_1, \cdots, U_n$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leqq 1$. A space X is said to be *chainable* if each open covering of X has a chain refinement. Hence a chainable space is a continuum.

If X is a metric space and $U_1, \cdots, U_n$ is a chain covering of X such that for some $\varepsilon > 0$, diameter $U_i < \varepsilon$ for $i = 1, \cdots, n$, then the chain $U_1, \cdots, U_n$ is said to be an *ε-chain covering* of X. A metric space X is *snakelike* if for each $\varepsilon > 0$, there exists an ε-chain covering of X.

2. Preliminary results. The following basic results will be needed. When proofs are omitted, they may be found in the references as indicated.

2—1. *Let $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ be an inverse system of compact metric spaces, where Λ is countable. Then X_∞ is a metric space.*

Proof. Since Λ is countable, we may choose a countable, linearly ordered cofinal subset Λ' of Λ, such that if $\lambda_i, \lambda_j \in \Lambda'$ and $i < j$, then $\lambda_i < \lambda_j$. Let d_i be the diameter of X_{λ_i}. Then a metric for $P_{\lambda_i \in \Lambda'} X_{\lambda_i}$ is defined as follows: For $\{x_{\lambda_i}\}$ and $\{y_{\lambda_i}\} \in P_{\lambda_i \in \Lambda'} X_{\lambda_i}$, set

$$\rho(\{x_{\lambda_i}\}, \{y_{\lambda_i}\}) = \sum_{i=1}^{\infty} 2^{-i} d_i^{-1} \rho_i(x_{\lambda_i}, y_{\lambda_i})$$

where ρ_i is the metric on X_{λ_i}. Then since ρ is a metric on $P_{\lambda_i \in \Lambda'} X_{\lambda_i}$, it is also a metric for X'_∞. Thus X_∞ is a metric space, since X_∞ is homeomorphic to X'_∞ by [2, 2.11].

2—2. *Let $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ be an inverse system of continua, with limit space X_∞. If X_λ is chainable for each $\lambda \in \Lambda$, then X_∞ is chainable.*

Proof. X_∞ is a continuum by [2, 2.5 and 2.10]. Mardesic [4] has shown that X_∞ is chainable if each $f_{\lambda\mu}$ is onto. We use this result here.

Let $A_\lambda = \Pi_\lambda(X_\infty)$ and $g_{\lambda\mu} = f_{\lambda\mu} \mid A_\lambda$. Then by [2, 2.8], $\{A_\lambda, g_{\lambda\mu}, \Lambda\}$ is an inverse system, each $g_{\lambda\mu}$ is onto and the limit space A_∞ of this system is X_∞. Since each subcontinuum of a chainable continuum is chainable, A_λ is chainable for each $\lambda \in \Lambda$. Thus $A_\infty = X_\infty$ is chainable by [4].

3. Inverse limits of indecomposable continua.

3—1. THEOREM. *Let $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ be an inverse limit system of indecomposable continua, where each function $f_{\lambda\mu}$ is onto. Then the inverse limit space X_∞ is an indecomposable continuum.*

Proof. X_∞ is a continuum by [2, 2.5 and 2.10]. Suppose X_∞ is decomposable, i.e., suppose there exist proper subcontinua H and K of X_∞ such that $X_\infty = H \cup K$.

We show first that there exists $\gamma \in \Lambda$ such that $\Pi_\gamma(H) \subsetneqq X_\gamma$. If not, then for all $\lambda \in \Lambda$, $\Pi_\lambda(H) = X_\lambda$. Let $\{x_\lambda\} \in X_\infty$ such that $\{x_\lambda\} \notin H$, and let N be any neighborhood of $\{x_\lambda\}$. Then there exist indices λ_i, $i = 1, 2, \cdots n$ and neighborhoods N_{λ_i} of each $x_{\lambda_i} \in X_{\lambda_i}$ such that $N = \{\{y_\lambda\} \in X_\infty \mid y_{\lambda_i} \in N_{\lambda_i}, i = 1, 2, \cdots, n\}$. Since Λ is a directed set, there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 > \lambda_i$, $i = 1, 2, \cdots, n$. Let $U_{\lambda_0} = \bigcap_{i=1}^n f_{\lambda_0\lambda_i}^{-1}(N_{\lambda_i})$. Then U_{λ_0} is an open subset of X_{λ_0} and $N = \{\{y_\lambda\} \in X_\infty \mid y_{\lambda_0} \in U_{\lambda_0}\}$. Now since $\Pi_\lambda(H) = X_\lambda$ for all $\lambda \in \Lambda$, there exists a point $\{x'_\lambda\} \in H$ such that $\Pi_{\lambda_0}(\{x'_\lambda\}) \in U_{\lambda_0}$, and hence $\{x'_\lambda\} \in N$. Thus $\{x_\lambda\}$ is a limit point of H. This is a contradiction, since H is closed and $\{x_\lambda\} \notin H$. Thus there exists $\gamma \in \Lambda$ such that $\Pi_\gamma(H) \subsetneqq X_\gamma$.

Similarly, there exists $\beta \in \Lambda$ such that $\Pi_{\beta}(K) \subsetneqq X_{\beta}$. Since Λ is a directed set, there exists $\delta \in \Lambda$ such that $\delta > \beta$ and $\delta > \gamma$. We show that $\Pi_{\delta}(H) \subsetneqq X_{\delta}$. For if $\Pi_{\delta}(H) = X_{\delta}$, then $\Pi_{\gamma}(H) = f_{\delta\gamma}(X_{\delta})$ since $f_{\delta\gamma}\Pi_{\delta} = \Pi_{\gamma}$. But $f_{\delta\gamma}$ is onto and hence $f_{\delta\gamma}(X_{\delta}) = X_{\gamma}$. Thus we have $\Pi_{\gamma}(H) = X_{\gamma}$, a contradiction. Therefore, $\Pi_{\delta}(H) \subsetneqq X_{\delta}$, and similarly $\Pi_{\delta}(K) \subsetneqq X_{\delta}$.

Now since Π_{δ} is continuous, $\Pi_{\delta}(H)$ and $\Pi_{\delta}(K)$ are subcontinua of X_{δ}. Also, $\Pi_{\delta}(X_{\infty}) = X_{\delta}$ [2, 2.6]. Therefore

$$X_{\delta} = \Pi_{\delta}(X_{\infty}) = \Pi_{\delta}(H \cup K) = \Pi_{\delta}(H) \cup \Pi_{\delta}(K) .$$

This is a contradiction, since X_{δ} is indecomposable.

3—2. THEOREM. *Let $\{X_{\lambda}, f_{\lambda\mu}, \Lambda\}$ be an inverse limit system of hereditarily indecomposable continua. Then the limit space X_{∞} is hereditarily indecomposable.*

Proof. X_{∞} is a continuum by [2, 2.5 and 2.10]. Let M be any subcontinuum of X_{∞}. We show that M is indecomposable.

Let $M_{\lambda} = \Pi_{\lambda}(M)$ and let $g_{\lambda\mu} = f_{\lambda\mu} \mid M_{\lambda}$. Each M_{λ} is a subcontinuum of X_{λ} and hence indecomposable. Also, by [2, 2. 8], $\{M_{\lambda}, g_{\lambda\mu}, \Lambda\}$ is an inverse system, each $g_{\lambda\mu}$ is onto and the limit space M_{∞} of this system is M. Thus M is indecomposable by Theorem 3—1.

3.—3. COROLLARY. *Let $\{X_{\lambda}, f_{\lambda\mu}, \Lambda\}$ be an inverse limit system of hereditarily indecomposable continua. Then the inverse limit space X_{∞} is an indecomposable continuum.*

Corollary 3—3 shows that Theorem 3—1 remains valid when the functions $f_{\lambda\mu}$ are not onto, provided that each X_{λ} is hereditarily indecomposable.

3—4. THEOREM. *Let $\{X_{\lambda}, f_{\lambda\mu}, \Lambda\}$ be an inverse limit system of pseudo-arcs. Let X_{∞} be the inverse limit space. Then X_{∞} is a chainable, hereditarily indecomposable continuum. If Λ is countable and X_{∞} is nondegenerate, then X_{∞} is a pseudo-arc.*

Proof. X_{∞} is a hereditarily indecomposable continuum by Theorem 3—2. For metric spaces, the definitions of chainable and snakelike continua are equivalent. Thus each X_{λ} is chainable and hence X_{∞} is chainable by 2—2.

If Λ is countable, then X_{∞} is a metric space by 2—1, and thus snakelike. Therefore, X_{∞} is a hereditarily indecomposable snakelike continuum, and hence a pseudo-arc if it is nondegenerate [1].

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