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A LOCALLY CONVEX TOPOLOGY ON A PREORDERED SPACE

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The purpose of this paper is to introduce a locally convex topology $\mathscr{T}_c$ on a preordered topological space $(X, \mathscr{T})$ in such a way that, if $\mathscr{T}_c$ is weaker than $\mathscr{T}$, then it is the l.u.b. of all locally convex topologies weaker than $\mathscr{T}$. Some of the consequences of having such a topology defined are examined, and the concepts of c-continuity and c-limit of a function are introduced. As an application of the machinery developed, a theorem concerning the unique extendability of functions from dense subsets of preordered spaces into regularly preordered spaces is established.

The terminology, with respect to convexity and order, is primarily that of [2]. For example, $i(A)(d(A))$ is the *smallest increasing* (*decreasing*) subset which contains the subset A. By definition, $x \in i(A)$ means that there is an $a \in A$ with $a < x$, where $<$ is the given preorder (called a *quasi-order* in [4]). Known results concerning general topological structures and filters are contained in [1]. Also, consistent with [1], the *Hausdorff Separation Axiom* is implied by *regularity*.

2. Convexity and c-continuity. Let $(X, \mathscr{T})$ be a preordered topological space with topology $\mathscr{T}$, and let $\mathscr{O}_c$ be the class of all subsets of X of the form $i(U)$ or $d(U)$, where U is a $\mathscr{T}$-open subset of X. Then $\mathscr{O}_c$ *generates* (i.e., is a *subbase* for) a convex topology $\mathscr{T}_c$ on X. We call $\mathscr{T}_c$ *the associated convex topology on* X.

LEMMA 1. *The following three conditions are equivalent in a preordered space* $(X, \mathscr{T})$:

(*) *For every increasing* (*decreasing*) *subset A of X, the interior, A°, of A is increasing* (*decreasing*).

(**) *For every decreasing* (*increasing*) *subset A of X, the closure, A^-, of A is decreasing* (*increasing*).

(***) *For every open set U, both $i(U)$ and $d(U)$ are open.*

Proof. (*) is clearly equivalent to (**). If (*) holds and U is open, then $i(U)$ increasing implies $[i(U)]^\circ$ increasing. It follows that $[i(U)]^\circ = i(U)$, so that (***) holds. On the other hand, if (***) holds and A is any increasing subset of X, then $i(A^\circ) = A^\circ$ and, therefore, (*) follows. The decreasing case is similarly established.

A preordered space $(X, \mathscr{T})$ in which condition (*) holds will be called a **-space*.

THEOREM 1. *Let $(X, \mathscr{T})$ be a preordered space. Then $\mathscr{T}_c$ is weaker than $\mathscr{T}$ (written $\mathscr{T}_c \leqq \mathscr{T}$) if and only if $(X, \mathscr{T})$ is a *-space. In this case, $(X, \mathscr{T}_c)$ is also a *-space. If $(X, \mathscr{T})$ is a locally convex space, then $\mathscr{T} \leqq \mathscr{T}_c$. In any case, with respect to the preorder on X, if $\mathscr{F}$ is any locally convex topology on X weaker than $\mathscr{T}$, then $\mathscr{F} \leqq \mathscr{T}_c$.*

Proof. It follows from (***) that $(X, \mathscr{T})$ is a *-space if and only if every set of $\mathscr{O}_c$ is $\mathscr{T}$-open. Hence we get the first statement. The second statement is simply a consequence of $\mathscr{T}_c \leqq \mathscr{T}$ and the definition of $\mathscr{T}_c$. If $\mathscr{T}$ is a locally convex topology and U is any convex $\mathscr{T}$-neighborhood of a point $x \in X$, then $i(U^\circ) \cap d(U^\circ)$ is a convex $\mathscr{T}_c$-neighborhood of x contained in U. Hence $\mathscr{T} \leqq \mathscr{T}_c$. Finally, if U is a convex $\mathscr{F}$-neighborhood of $x \in X$, then there is a $\mathscr{T}$-open neighborhood V of x contained in U. Hence $i(V) \cap d(V) \subset U$ implies that U is a $\mathscr{T}_c$-neighborhood of x. Hence $\mathscr{F} \leqq \mathscr{T}_c$.

Let $(X, \mathscr{T})$ be a preordered space and $A \subset X$. A is called a *subspace* of X when A is given both the preorder and the topology $\mathscr{T}(A)$ induced from $(X, \mathscr{T})$. A function f from $(X, \mathscr{T})$ into a preordered space $(Y, \mathscr{F})$ is said to be *c-continuous* if f is continuous from $(X, \mathscr{T}_c)$ into $(Y, \mathscr{F}_c)$. Thus, if $y \in X$ and $z \in Y$, then z is a *c-limit of f* if the image, $f(\mathscr{V}_c(y))$, of the filter of all $\mathscr{T}_c$-neighborhoods of y $\mathscr{F}_c$-converges to z. In this case we write $z = c\text{-}\lim_{x \to y} f(x)$. Consistent with this, if $y \in A^-$, then $z = c\text{-}\lim_{x \to y} (A) f(x)$ means that the filter base $f(\mathscr{V}_c(y) \cap A)$ $\mathscr{F}_c$-converges to z, where $\mathscr{V}_c(y) \cap A$ is the *trace* on A of $\mathscr{V}_c(y)$.

As a result of the above, we have:

LEMMA 2. *A function f from a preordered space $(X, \mathscr{T})$ into a preordered space $(Y, \mathscr{F})$ is c-continuous at a point $y \in X$ if and only if $f(y) = c\text{-}\lim_{x \to y} f(x)$. If both $\mathscr{T}$ and $\mathscr{F}$ are locally convex *-topologies, then c-continuity and continuity are equivalent.*

3. Regularly preordered spaces. Let A be a subset of a preordered space $(X, \mathscr{T})$. By $I(A)$ $(D(A))$ we mean the *closed increasing (decreasing) hull* of A in X. [2].

LEMMA 3. *The following two conditions on a preordered space $(X, \mathscr{T})$ are equivalent:*

(MR) *If F is a closed increasing (decreasing) subset of X and*

$x \notin F$, then there exist disjoint open neighborhoods U of x and V of F such that U is decreasing (increasing) and V is increasing (decreasing).

(MR_1) *If $x \in X$ and U is an open decreasing (increasing) neighborhood of x, then there is an open decreasing (increasing) neighborhood V of x such that $D(V) \subset U$ $(I(V) \subset U)$.*

Proof. Suppose (MR) holds and let U be an open decreasing neighborhood of x. Then we get disjoint open sets V and W such that $x \in V$, V is decreasing, $X \backslash U \subset W$, and W is increasing (where $X \backslash U$ is the *complement* of U in X). Hence $V \subset D(V) \subset X \backslash W \subset U$, as required for ($\mathrm{MR}_1$). Conversely, if ($\mathrm{MR}_1$) holds and F is a closed increasing set with $x \notin F$, then there is an open decreasing neighborbood V of x with $D(V) \subset X \backslash F$. Hence V and $X \backslash D(V)$ are the neighborhoods required for (MR).

In accordance with [3], condition (MR) is called *monotone regularity*. A space $(X, \mathscr{T})$ equipped with a *closed* (or equivalently, *continuous*) preorder and satisfying (MR) is called a *regularly preordered space.*

If the order of X is *discrete* (i.e., $x < y$ means $x = y$), then $(X, \mathscr{T})$ will be a regularly ordered space if and only if it is regular (where it is shown in [2], that a space with a closed order is a Hausdorff space).

LEMMA 4. *If $(X, \mathscr{T})$ is a preordered *-space, then condition (MR_1) is equivelent to the following:*

(MR_2) *For every $x \in X$ and any decreasing (increasing) neighborhood V of x, there is an open decreasing (increasing) neighborhood W of x such that $D(W) \subset V$ $(I(W) \subset V)$.*

COROLLARY. *In a regularly preordered *-space $(X, \mathscr{T})$, every convex neighborhood of a point $x \in X$ contains a closed convex neighborhood of x.*

EXAMPLE 1. Let X be the set of real numbers with its usual order. Define a topology $\mathscr{T}$ on X by requiring that a subset V of X be a neighborhood of a point $x \in X$ if and only if there is a real $\varepsilon > 0$ such that V contains the open interval $(x - \varepsilon, x + \varepsilon)$ if x is irrational, and V contains $(x - \varepsilon, x + \varepsilon) \cap P$ if $x \in P$, where P is the set of rationals. Then $(X, \mathscr{T})$ is a regularly ordered *-space which is not regular. Moreover $\mathscr{T}_c(P)$, $\mathscr{T}(P)$, and $\mathscr{T}(P)_c$ are all equal. On the other hand, if A is the unit interval $[0, 1]$, then $\mathscr{T}_c(A) < \mathscr{T}(A)$ and $\mathscr{T}_c(A) = \mathscr{T}(A)_c$.

EXAMPLE 2. Let N be the set of integers with its usual order and $\mathcal{T}$ the *topology of finite complements* on N. Then $(N, \mathcal{T})$ is an ordered *-space with $\mathcal{T}(A)_c = \mathcal{T}(A)$ and $\mathcal{T}_c(A) < \mathcal{T}(A)_c$ for every finite subset $A \subset N$ containing more than one element.

THEOREM 2. *If $(X, \mathcal{T})$ is a regularly ordered *-space, then $(X, \mathcal{T}_c)$ is regular. Consequently, $\mathcal{T}_c$ is the strongest regular locally convex topology on X weaker than $\mathcal{T}$.*

Proof. To see that $\mathcal{T}_c$ is a Hausdorff topology, let x and y be distinct points of X. Since X is ordered, we can assume that $x \not< y$. Now, $F = d(y)$ is closed, decreasing, and $x \notin F$. Hence (MR) guarantees the separation of x and y by $\mathcal{T}$-open neighborhoods U of x and V of y such that U is increasing and V is decreasing. It follows that x and y are separated by $\mathcal{T}_c$-neighborhoods. Suppose now that F is a $\mathcal{T}_c$-closed set and $x \notin F$. Then there exist $\mathcal{T}_c$-open neighborhoods U and V of x with U decreasing, V increasing, and $U \cap V \cap F = \phi$. Since $\mathcal{T}_c \leqq \mathcal{T}$, it follows from ($\mathrm{MR}_2$) that there is a $\mathcal{T}$-open neighborhood W of x such that $D(W) \cap I(W) \subset U \cap V$. Let $U_0 = i(W) \cap d(W)$, and let V_0 be the complement of the $\mathcal{T}_c$-closure of $D(W) \cap I(W)$. Then $U_0 \cap V_0 = \phi$; so $(X, \mathcal{T}_c)$ is regular.

4. *c*-continuous extensions. Let A be a subset of a preordered space $(X, \mathcal{T})$ and U an open subset of X such that $U \cap A \neq \phi$. Then A is said to be *scattered with respect to* U if $i(U \cap A) \cap A = i(U) \cap A$ and $d(U \cap A) \cap A = d(U) \cap A$. If A is scattered with respect to every open set which it meets, then A is said to be *scattered*.

If A is dense and scattered, then $\mathcal{T}_c(A) = \mathcal{T}(A)_c$. If a scattered subset A is not dense, then $\mathcal{T}_c(A) = \mathcal{T}(A)_c$ whenever $(X, \mathcal{T})$ is a *-space. On the other hand, Example 2 yields instances of nonscattered subsets.

THEOREM 3. *Let $(X, \mathcal{T})$ be a preordered space, A a $\mathcal{T}$-dense subset of X, and f a c-continuous function from $(A, \mathcal{T}(A))$ into a regularly ordered *-space $(Y, \mathcal{F})$. Then there exists a unique c-continuous extension g of f to $(X, \mathcal{T})$ if and only if the* $c\text{-}\lim_{x \to y}(A)f(x)$ *exists for every $y \in X$. If A is scattered, then f order-preserving implies g order-preserving.*

Proof. We note that A $\mathcal{T}$-dense implies A $\mathcal{T}_c$-dense. Now, if the extension g exists, then its continuity from $(X, \mathcal{T}_c)$ into $(Y, \mathcal{F}_c)$ guarantees the existence of the $c\text{-}\lim_{x \to y}(A)f(X)$ for every $y \in X$. [1]. Suppose that the c-limit exists for every $y \in X$ and let $a \in A$, $\mathcal{U}$ the filter on Y generated by the class $f(\mathscr{V}_c(a) \cap A) \cup f(\mathscr{V}_A(a)_c)$, where $\mathscr{V}_A(a)_c$ is the filter of all $\mathscr{T}(A)_c$-neighborhoods of a. By hypothesis, $z = c\text{-}\lim(A)_{x \to a} f(x)$ for some $z \in Y$; so $\mathscr{U}$ is stronger than $\mathscr{V}_c(z)$. By the c-continuity of f on $(A, \mathscr{T}(A))$, $\mathscr{U}$ is stronger than $\mathscr{V}_c(f(a))$. It follows that $z = f(a)$ since $(Y, \mathscr{F}_c)$ is a Hausdorff space. Hence f is continuous from $(A, \mathscr{T}_c(A))$ into $(Y, \mathscr{F}_c)$ and, therefore, there is a unique extension g of f which is continuous from $(X, \mathscr{T}_c)$ into $(Y, \mathscr{F}_c)$ [1]. Hence g is the required extension of f.

Let A be scattered, f order-preserving, and $x < y$. It remains to show that $g(x) < g(y)$. Let $\mathscr{B}$ be the class of all sets of the form $U \cap A$, where U is a decreasing $\mathscr{T}_c$-neighborhood of y. Then $\mathscr{B}$ is filter base on X such that x is a $\mathscr{T}_c$-adherent point of $\mathscr{B}$. Since g is continuous, $g(x)$ is an $\mathscr{F}_c$-adherent point of $g(\mathscr{B}) = f(\mathscr{B})$. Suppose that $g(x) \nless g(y)$. Then there exist $\mathscr{F}_c$-open neighborhoods C of $g(x)$ and D of $g(y)$ such that C is increasing, D is decreasing, and $C \cap D = \phi$. Let $V = g^{-1}(D)$, so that $d(V) \cap A \in \mathscr{B}$. Since A is scattered, $d(V) \cap A = d(V \cap A) \cap A$. If $v \in d(V \cap A) \cap A$, then there is a $u \in V \cap A$ with $v < u$. Hence $g(v) = f(v) < f(u) = g(u) \in D$ implies $g(v) \in D$ and, therefore, $v \in V \cap A$. That is, $d(V) \cap A = V \cap A$. But $f(V \cap A) \cap C \subset D \cap C = \phi$ contradicts the adherence of $g(x)$ to $f(\mathscr{B})$. Consequently we must have $g(x) < g(y)$.

THEOREM 4. *Let f be an order-preserving function from a subspace A of a preordered space $(X, \mathscr{T})$ into a *-space $(Y, \mathscr{F})$. If f has a continuous order-preserving extension g to X, then g is c-continuous.*

Proof. Let C be a convex $\mathscr{F}_c$-neighborhood of $g(x)$ in Y. Then $\mathscr{F}_c \leqq \mathscr{F}$ and g continuous yield an open neighborhood V of x in X with $g(V) \subset C$. Thus $U = i(V) \cap d(V)$ is a convex $\mathscr{T}_c$-neighborhood of x and, since g is increasing, $g(U) \subset C$. Hence g is continuous from $(X, \mathscr{T}_c)$ into $(Y, \mathscr{F}_c)$ and, therefore, g is c-continuous on X.

BIBLIOGRAPHY

1. N. Bourbaki, *Topologie générale*. Livre III: *Structures topologique*, Actualités Sci. Ind. No. 1142, Hermann, Paris, 1961.
2. L. Nachbin, *Topology and order*, Van Nostrand Math. Studies D. Van Nostrand Co., Inc., New York, New York, 1964.
3. L. E. Ward Jr., *Binary relations in topological spaces*, An. Acad. Brasil. Ci. **26** (1954), 357-373.
4. L. E. Ward Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. **5** (1954), 144-166.

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