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POWER SERIES RINGS OVER A KRULL DOMAIN

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Let D be a Krull domain and let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a set of indeterminates over D. This paper shows that each of three "rings of formal power series in $\{X_\lambda\}$ over D" are also Krull domains; also, some relations between the structure of the set of minimal prime ideals of D and the set of minimal prime ideals of these rings of formal power series are established.

In considering formal power series in the X_λ's over D, there are three rings which arise in the literature and which are of importance. We denote these here by $D[[\{x_\lambda\}]]_1$, $D[[\{X_\lambda\}]]_2$, and $D[[\{X_\lambda\}]]_3$. $D[[\{X_\lambda\}]]_1$ arises in a way analogous to that of $D[\{X_\lambda\}]$—namely, $D[[\{X_\lambda\}]]$ is defined to be $\bigcup_{F \in \mathscr{F}} D[[F]]$, where $\mathscr{F}$ is the family of all finite nonempty subsets of Λ. $D[[\{X_\lambda\}]]_2$ is defined to be

$$\left\{\sum_{i=0}^{\infty} f_i \mid f_i \in D[\{X_\lambda\}],\ f_i = 0 \text{ or a form of degree } i\right\},$$

where equality, addition, and multiplication are defined on $D[[\{x_\lambda]]_2$ in the obvious ways. $D[[\{X_\lambda\}]]_2$ arises as the completion of $D[\{X_\lambda\}]$ under the $(\{X_\lambda\})$-adic topology; the topology on $D[[\{X_\lambda]]_2$ is induced by the decreasing sequence $\{A_i\}_0^\infty$ of ideals, where A_i consists of those formal power series of order $\geqq i$—that is, those of the form $\sum_{j=i}^{\infty} f_j$. If Λ is infinite, A_1 properly contains the ideal of $D[[\{X_\lambda\}]]_2$ generated by $\{X_\lambda\}$. Finally, $D[[\{X_\lambda\}]]_3$ is the *full* ring of formal power series over D, and is defined as follows (cf. [1, p. 66]): Let N be the set of nonnegative integers, considered as an additive abelian semigroup, and let S be the weak direct sum of N with itself $|\Lambda|$ times. S is an additive abelian semigroup with the property that for any $s \in S$, there are only finitely many pairs (t, u) of elements of S whose sum is s. $D[[\{X_\lambda\}]]_3$ is defined to be the set of all functions $f: S \rightarrow$ D, where $(f+g)(s) = f(s) + g(s)$ and where $(fg)(s) = \sum_{t+u=s} f(t)g(u)$ for any $s \in S$, the notation $\sum_{t+u=s}$ indicating that the sum is taken over all ordered pairs (t, u) of elements of S with sum s. To within isomorphism we have $D[[\{X_\lambda\}]]_1 \subseteqq D[[\{X_\lambda\}]]_2 \subseteq D[[\{X_\lambda\}]]_3$, and each of these containments is proper if and only if Λ is infinite. Our method of attack in showing that $D[[\{X_\lambda\}]]_i$, $i = 1, 2, 3$, is a Krull domain if D is consists in showing that $D[[\{X_\lambda\}]]_3$ is a Krull domain and that $D[[\{X_\lambda\}]]_3 \cap K_i = D[[\{X_\lambda\}]]_i$ for $i = 1, 2$, where K_i denotes the quotient field of $D[[\{X_\lambda\}]]_i$.

1. **The proof that $D[[\{X_\lambda\}]]_3$ is a Krull domain.** Using the

notation of the previous section, we introduce some terminology which will be helpful in showing that $D[[\{X_\lambda\}]]_3$ is a Krull domain. We think of the elements of S as $|\Lambda|$-tuples $\{n_\lambda\}_{\lambda\in\Lambda}$ which are finitely nonzero. For $s=\{n_\lambda\}\in S$, we define $\pi(s)$ to be $\sum_{\lambda\in\Lambda}n_\lambda$ and we denote by S_i the set of elements s of S such that $\pi(s)=i$; clearly π is a homomorphism from S onto N. Given a well-ordering on the set Λ, we well-order the set S as follows: if $s=\{m_\lambda\}$ and $t=\{n_\lambda\}$ are distinct elements of S, then $s<t$ if $\pi(s)<\pi(t)$ or if $\pi(s)=\pi(t)$ and $m_\lambda<n_\lambda$ for the first λ in Λ such that m_λ and n_λ are unequal. It is clear that this ordering on S is compatible with the semigroup operation—that is, $s_1<s_2$ implies that $s_1+t<s_2+t$ for any t in S. Also, S is cancellative and $s_1+t<s_2+t$ implies that $s_1<s_2$.

If $f\in D[[\{X_\lambda\}]]_3-\{0\}$, we say that f is a *form of degree* i, where $i\in N$, provided f vanishes on $S-S_i$; the *order of* f, denoted by $\mathcal{O}(f)$, is defined to be the smallest nonnegative integer t such that f does not vanish on S_t. If $\mathcal{O}(f)=k$, then the *initial form of* f is defined to be that element f_k of $D[[\{X_\lambda\}]]_3$ which agrees with f on S_k and which vanishes on $S-S_k$.

LEMMA 1.1. *If* $f, g\in D[[\{X_\lambda\}]]_3-\{0\}$, *then*

(1) *If* $f+g\neq 0$, $\mathcal{O}(f+g)\geqq\min\{\mathcal{O}(f), \mathcal{O}(g)\}$.

(2) $\mathcal{O}(fg)=\mathcal{O}(f)+\mathcal{O}(g)$.

(3) *If f and g are forms of degree m and n, respectively, then fg is a form of degree $m+n$.*

(4) *The initial form of fg is the product of the initial forms of f and of g.*

Proof. In a less general context, Lemma 1.1 is a well known result; we prove only (2) and (3) here.

(2): We let s be the smallest element of S on which f does not vanish and we let t be the corresponding element for g. By definition of π and $\mathcal{O}$, $\pi(s)=\mathcal{O}(f)=i$ and $\pi(t)=\mathcal{O}(g)=j$. To show that $\mathcal{O}(fg)=i+j$, we prove that $(fg)(s+t)\neq 0$ and that $(fg)(u)=0$ for $u<s+t$. The second statement is clear, for if $s'+t'=u$, then either $s'<s$ or $t'<t$ so that $f(s')=0$ or $g(t')=0$ and $f(s')g(t')=0$ in either case. By similar reasoning, we see that $(fg)(s+t)=f(s)g(t)\neq 0$. Hence $\mathcal{O}(fg)=i+j$.

(3): By (2), $\mathcal{O}(fg)=m+n$. To see that fg is a form, we need only observe that fg vanishes on S_k for any $k>m+n$. Thus if $w\in S_k$, then $(fg)(w)=\sum_{u+v=w}f(u)g(v)$ and for each such pair (u, v) either $\pi(u)>m$ or $\pi(v)>n$ so that $f(u)=0$ or $g(v)=0$ so that $(fg)(w)=\sum_{u+v=w}f(u)g(v)=0$.

LEMMA 1.2. *Let K be a field and let $\{D_\alpha\}$ be a family of sub-*

domains of K such that each D_α is a Krull domain. Let $D = \bigcap_\alpha D_\alpha$ and suppose that each nonzero element of D is a nonunit in only finitely many D'_αs. Then D is a Krull domain.

Proof. For each α we consider a defining family $\{V_\beta^{(\alpha)}\}$ of rank one discrete valuation rings for D_α. If L is the quotient field of D and $\mathscr{S} = \{V_\beta^{(\alpha)} \cap L\}_{\alpha,\beta}$, $\mathscr{S}$ is a family of discrete valuation rings of rank $\leqq 1$, and the intersection of the members of the collections $\mathscr{S}$ is D. If d is a nonzero element of D, then d is a nonunit in only finitely many D'_αs, say $D_{\alpha_1}, \cdots, D_{\alpha_n}$. Because D_{α_i} is a Krull domain and $\{V_\beta^{(\alpha_i)}\}$ is a defining family for D_{α_i}, d is a nonunit in only finitely many of the $V_\beta^{(\alpha_i)}$'s. Therefore D is a Krull domain and the family of essential valuations for D is a subfamily of $\{V_\beta^{(\alpha)} \cap L\}_{\alpha,\beta}$ [6, p. 116].

We now give an outline of our proof that $D[[\{X_\lambda\}]]_3$ is a Krull domain when D is a Krull domain. Let K be the quotient field of D and let $\{V_\alpha\}$ be the family of essential valuation rings for D [7, p. 82]. By a result due to Cashwell and Everett [3] (see also [4]), $J[[\{X_\lambda\}]]_3$ is a unique factorization domain (UFD), where J is an integral domain with identity, if and only if $J[[Y_1, \cdots, Y_n]]$ is a UFD for any positive integer n. If J is a principal ideal domain, then $J[[Y_1, \cdots, Y_n]]$ is a UFD for any n [2, pp. 42, 100]; in particular, $K[[\{X_\lambda\}]]_3$ and $V_\alpha[[\{x_\lambda\}]]_3$ are then UFD's for each α. Consequently, $(V_\alpha[[\{X_\lambda\}]]_3)_{N_\alpha}$ is a UFD for any multiplicative system N_α in $V_\alpha[[\{X_\lambda\}]]_3$. To show that $D[[\{X_\lambda\}]]_3$ is a Krull domain, it will be sufficient, in view of Lemma 1.2, to show that by appropriate choices of the multiplicative systems N_α, we can express $D[[\{X_\lambda\}]]_3$ as

$$K[[\{X_\lambda\}]]_3 \cap (\textstyle\bigcap_\alpha (V_\alpha[[\{X_\lambda\}]]_3)_{N_\alpha}),$$

where each nonzero element of $D[[\{X_\lambda\}]]_3$ is a nonunit in only finitely many $(V_\alpha[[\{X_\lambda\}]]_3)_{N_\alpha}$'s. We define N_α as follows:
$N_\alpha = \{f \in V_\alpha[[\{X_\lambda\}]]_3 - \{0\} \mid \mathscr{O}(f) = i$ and there exists $s \in S_i$ such that $f(s)$ is a unit of $V_\alpha\}$, and we prove

PROPOSITION 1.3. *N_α is a multiplicative system in $V_\alpha[[\{X_\lambda\}]]_3$.*

$$(V_\alpha[[\{X_\lambda\}]]_3)_{N_\alpha} \cap K[[\{X_\lambda\}]]_3 = V_\alpha[[\{X_\lambda\}]]_3\ ,$$

so that

$$D[[\{X_\lambda\}]]_3 = K[[\{X_\lambda\}]]_3 \cap (\textstyle\bigcap_\alpha (V_\alpha[[\{X_\lambda\}]]_3)_{N_\alpha})\ .$$

Each nonzero element of $D[[\{X_\lambda\}]]_3$ is in all but a finite number of the N_α's.

Before giving the proof of Proposition 1.2, we recall a result concerning the content of the product of two polynomials. Let J be an integral domain with identity having quotient field F and for $f \in F[\{X_\lambda\}]$, let A_f denote the fractional ideal of J generated by the set of coefficients of f. In order that $A_{fg} = A_f A_g$ for each pair f, g of elements of $F[\{X_\lambda\}]$, it is necessary and sufficient that J be a Prüfer domain[1] [5, Th. 1]. In particular $A_{fg} = A_f A_g$ for each $f, g \in F[\{X_\lambda\}]$ if J is a valuation ring.

Proof of Proposition 1.3. To show that N_α is a multiplicative system, let $f, g \in N_\alpha$. Then the initial forms f_i, g_j of f and g are in N_α. $f_i g_j$ is the initial form of fg and $\mathcal{O}(fg) = i + j = \mathcal{O}(f) + \mathcal{O}(g)$. Therefore we need only show that $(fg)(s)$ is a unit of V_α for some $s \in S_{i+j}$. The smallest element u of S for which $f(u)$ is a unit in V_α is an element of S_i and the smallest element v of S for which $g(v)$ is a unit of V_α is an element of S_j. $u + v \in S_{i+j}$ and $(fg)(u + v) = \sum_{u'+v'=u+v} f(u')g(v')$ is a unit of V_α. For if $u' + v' = u + v$ and if $\{u', v'\} \neq \{u, v\}$, then either $u' < u$ or $v' < v$ so that $f(u')$ or $g(v')$, and hence $f(u')g(v')$, is a nonunit of V_α. It follows that $(fg)(u + v)$ is the unit $f(u)g(v)$ plus a nonunit of V_α. Therefore $(fg)(u + v)$ is a unit of V_α, $fg \in N_\alpha$, and N_α is a multiplicative system.

To prove that $K[[\{x_\lambda\}]]_3 \cap (V_\alpha[[\{x_\lambda\}]]_3)_{N\alpha} \subseteqq V_\alpha[[\{X_\lambda\}]]_3$, (the opposite containment is clear), we must show that if $f \in K[[\{X_\lambda\}[]_3 - \{0\}$ and if there is an element g of N_α such that $fg \in V_\alpha[[\{X_\lambda\}]]_3$, then $f \in V_\alpha[[\{X_\lambda\}]]_3$. By induction, it suffices to show that the initial form f_i of f is in $V_\alpha[[\{X_\lambda\}]]_3$. If g_j is the initial form of g, then $g_j \in N_\lambda$ and $f_i g_j$, the initial form of fg, is in $V_\alpha[[\{X_\lambda\}]]_3$. We can therefore assume without loss of generality that f and g are forms of degree i and j, respectively. Let $s \in S_i$. We must show that $f(s) \in V_\alpha$. Let t be an element of S_j such that $g(t)$ is a unit of V_α. If $s = \{m_\lambda\}$ and if $t = \{n_\lambda\}$ there are only finitely many elements τ of Λ such that $m_\tau \neq 0$ or $n_\tau \neq 0$; let $\lambda_1, \lambda_2, \cdots, \lambda_u$ be this finite set of elements of Λ. There are only finitely many elements $\{k_\lambda\}$ of S_i such that $k_z = 0$ for each $z \notin [\lambda_1, \cdots, \lambda_u\}$; let these elements be $s_1, s_2, \cdots, s_p$. Also, there are only finitely many elements $\{k_\lambda\}$ of S_j such that $k_z = 0$ for each $z \notin \{\lambda_1, \cdots, \lambda_u\}$, and we let these elements be $t_1, t_2, \cdots, t_r$. If f^* is the polynomial $\sum_{q=1}^{p} f(s_q) X_{\lambda_1}^{n_{\lambda_1}^{(q)}} \cdots X_{\lambda_u}^{n_{\lambda_u}^{(q)}}$, where $s_q = \{n_\lambda^{(q)}\}$ and if $g^* = \sum_{q=1}^{r} g(t_q) X_{\lambda_1}^{m_{\lambda_1}^{(q)}} \cdots X_{\lambda_u}^{m_{\lambda_u}^{(q)}}$, where $t_q = \{m_\lambda^{(q)}\}$, then by definition of addition in S, it is true that $(fg)(\{k_\lambda\})$ is equal to the coefficient of $X_{\lambda_1}^{k_{\lambda_1}} \cdots X_{\lambda_u}^{k_{\lambda_u}}$ in f^*g^* for any $\{k_\lambda\}$ in S_{i+j} such that $k_\lambda = 0$ for $\lambda \notin \{\lambda_1, \cdots, \lambda_u\}$.

[1] A *Prüfer domain* is an integral domain with identity in which each nonzero finitely generated ideal is invertible.

Therefore, $f^*g^* \in V_\alpha[X_{\lambda_1}, \cdots, X_{\lambda_u}]$ since $fg \in V_\alpha[[\{X_\lambda\}]]_3$. Further, $A_{g^*} = V_\alpha$ since $t \in \{t_1, \cdots, t_r\}$ and since $g(t)$ is a unit of V_α. Therefore $A_{f^*}A_{g^*} = A_{f^*} = A_{f^*g^*} \subseteqq V_\alpha$. But $f(s) \in A_{f^*}$ since $s \in \{s_1, s_2, \cdots, s_p\}$. Hence $f(s) \in V_\alpha$ and our proof is complete.

Finally, if h is a nonzero element of $D[[X_\lambda\}]]_3$ of order i, then we choose $s \in S_i$ such that $h(s) \neq 0$. Since $\{V_\alpha\}$ is the family of essential valuation rings for the Krull domain D, $h(s)$ is a unit in all but a finite set $\{V_{\alpha_1}, \cdots, V_{\alpha_w}\}$ of the V_α's. Hence h is in each N_α save $N_{\alpha_1}, \cdots, N_{\alpha_w}$.

THEOREM 1.4. *If D is a Krull domain, then $D[[\{X_\lambda\}]]_3$ is also a Krull domain.*

2. The proofs that $D[[\{X_\lambda\}]]_1$ and $D[[\{X_\lambda\}]]_2$ are Krull domains. In view of Theorem 1.4, in order to show that D Krull implies that $D[[\{X_\lambda\}]]_i$, $i = 1, 2$, is Krull, it is sufficient to show that for any integral domain J with identity, $J[[\{X_\lambda\}]]_3 \cap K_i = J[[\{X_\lambda\}]]_i$, where K_i denotes the quotient field of $J[[\{X_\lambda\}]]_i$. Thus we need to show that if $f \in J[[\{X_\lambda\}]]_3 - \{0\}$ and if g is a nonzero element of $J[[\{X_\lambda\}]]_i - \{0\}$ such that $fg \in J[[\{X_\lambda\}]]_i$, then $f \in J[[\{X_\lambda\}]]_i$. We consider first the case when $i = 2$. By induction, it suffices to show that the initial form of f is in $J[[\{X_\lambda\}]]_2$, and since the product of the initial form of f and the initial form of g is the initial form of fg and is in $J[[\{X_\lambda\}]]_2$, we need consider only the case when f and g are forms of degrees i and j, respectively. Since fg and g are in $J[[\{x_\lambda\}]]_2$, there is a finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of Λ such that g vanishes on each element $\{n_\lambda\}$ of S_j for which $n_\lambda \neq 0$ for some λ in $\Lambda - \{\lambda_k\}_1^n$ and such that fg vanishes on each element $\{m_\lambda\}$ of S_{i+j} for which $m_\lambda \neq 0$ for some λ in $\Lambda - \{\lambda_k\}_1^n$. We observe that this implies that f vanishes on each element $\{p_\lambda\}$ of S_i such that $p_\lambda \neq 0$ for some $\lambda \notin \{\lambda_1, \cdots, \lambda_n\}$, for if this were not the case, then there would be a smallest element $p = \{p_\lambda\}$ of S_i with $p_\mu \neq 0$ for some $\mu \notin \{\lambda_1, \cdots, \lambda_n\}$ for which $f(p) \neq 0$. Then if $s = \{s_\lambda\}$ is the smallest element of S_j for which $g(s) \neq 0$, we observe that $(fg)(p + s) = f(p)g(s) \neq 0$ and that $p + s = \{p_\lambda + s_\lambda\}$, where $p_\mu + s_\mu \geqq p_\mu > 0$, contrary to the hypothesis on fg. We see that $(fg)(p + s) = f(p)g(s)$ as follows: If $p' + s' = p + s$ where $p' \in S_i$ and $s' \in S_j$, then $s' < s$ implies that $g(s') = 0$ so that $f(p')g(s') = 0$. On the other hand, if $s' > s$, then $p' < p$ so that $f(p') = 0$ if $p' = \{p'_\lambda\}$ and $p'_\lambda \neq 0$, while $g(s') = 0$ if $p'_\mu = 0$ since the μ-th coordinate of s' is then nonzero. Consequently, $(fg)(p + s) = f(p)f(s)$, and the contradiction which this equality implies shows that it is indeed the case that $f(\{p_\lambda\}) = 0$ for each $\{p_\lambda\}$ in S_i such that $p_\lambda \neq 0$ for some $\lambda \notin \{\lambda_1, \cdots, \lambda_n\}$. Hence $f \in J[[\{X_\lambda\}]]_2$ as we wished to show.

Our proof for $J[[\{X_\lambda\}]]_2$ shows that if the set $\{\lambda_1, \cdots, \lambda_n\}$ does not depend on i, as is the case if g and fg are in $J[[\{X_\lambda\}]]_1$, then each form f_i associated with f (that is, $f\cdot\chi_i$, where χ_i is the characteristic function of S_i) will also have the property that it vanishes on each element $\{s_\lambda\}$ of S_i such that $s_\lambda \neq 0$ for some $\lambda \notin \{\lambda_1, \cdots, \lambda_n\}$. Consequently, $f \in J[[\{X_\lambda\}]]_1$. We have proved

THEOREM 2.1. *If D is a Krull domain, then $D[[\{X_\lambda\}]]_2$ and $D[[\{X_\lambda\}]]_1$ are also Krull domains.*

3. **Minimal primes of $D[[\{X_\lambda\}]]_3$.** Our proofs of Lemma 1.2 and Proposition 1.3 show the following, in case D is a Krull domain with quotient field K. If L is the quotient field of $D[[\{X_\lambda\}]]_3$, then the set of essential valuation rings for $D[[\{X_\lambda\}]]_3$ is a subset of $\{W_\sigma \cap L\} \cup \{W_\beta^{(\alpha)} \cap L\}$, where $\{W_\sigma\}$ is the family of essential valuation rings for $K[[\{X_\lambda\}]]_3$ and where $[W_\beta^{(\alpha)}\}$ is the family of essential valuation rings for $(V_\alpha[[\{X_\lambda\}]]_3)_{N_\alpha}$; $\{V_\alpha\}$ the family of essential valuation rings for D. Let M_σ be the center of $W_\sigma \cap L$ on $D[[\{X_\lambda\}]]_3$ and let $M_\beta^{(\alpha)}$ be the center of $W_\beta^{(\alpha)} \cap L$ on $D[[\{X_\lambda\}]]_3$. Since $K \subset W_\sigma$, $M_\sigma \cap K = (0)$; in particular, $M_\sigma \cap D = (0)$. Further, V_α is clearly contained in $W_\beta^{(\alpha)} \cap L$ so that $W_\beta^{(\alpha)} \cap L = V_\alpha$ or $W_\alpha^{(\alpha)} \cap L = K$. In the first case $M_\beta^{(\alpha)} \cap D = P_\alpha$ where $V_\alpha = D_{P_\alpha}$, and in the second $M_\beta^{(\alpha)} \cap D = (0)$. Since $D[[\{X_\lambda\}]]_3$ is a Krull domain, the set of minimal primes of $D[[\{X_\lambda\}]]_3$ is a subset of $\{M_\sigma\} \cup \{M_\beta^{(\alpha)}\}$. Hence we have proved

LEMMA 3.1. *Each minimal prime of $D[[X_\lambda\}]]_3$ meets D either in zero or in minimal prime of D.*

Our main purpose in this section is to prove:

THEOREM 3.2. *If P_α is a minimal prime of D, there is a unique minimal prime of $D[[\{X_\lambda\}]]_3$ which meets D in P_α.*

Our proof of Theorem 3.2 proceeds as follows. Let v_α be a valuation associated with the valuation ring D_{P_α}. We observe that the function v_α^* defined on $D[[\{X_\lambda\}]]_3$ by $v_\alpha^*(f) = \min\{v_\alpha(f(s)) \mid s \in S\}$ induces a valuation on L, the quotient field of $D[[\{X_\lambda\}]]_3$. To prove this, let $f, g \in D[[\{X_\lambda\}]]_3$ and suppose that $v_\alpha((f+g)(t)) = v_\alpha^*(f+g)$. Since $v_\alpha(f(t) + g(t)) \geqq \min\{v_\alpha(f(t)), v_\alpha(g(t))\} \geqq \min\{v_\alpha^*(f), v_\alpha^*(g)\}$, it follows that $v_\alpha^*(f+g) \geqq \min\{v_\alpha^*(f), v_\alpha^*(g)\}$. Also, if s is the smallest element of S such that $v_\alpha(f(s)) = v_\alpha^*(f)$ and if u is the smallest element of S such that $v_\alpha(g(u)) = v_\alpha^*(g)$, then it is straightforward to show that

$$\begin{aligned} v_\alpha((fg)(s+u)) &= v_\alpha(f(s)) + v_\alpha(g(u)) = v_\alpha^*(f) + v_\alpha^*(g) \\ &= \min\{v_\alpha((fg)(t)) \mid t \in S\} = v_\alpha^*(fg)\,. \end{aligned}$$

We denote the extension of v_α^* to L by v_α^* also; it is clear that v_α and v_α^* have the same value group so that v_α^* is rank one discrete and is an extension of v_α to L. The center of v_α^* on $D[[\{X_\lambda\}]]_3$ is the prime ideal $Q_\alpha = \{f \mid f(s) \in P_\alpha \text{ for each } s \in S\}$; we next prove that $(D[[\{X_\lambda\}]]_3)_{Q_\alpha}$ is the valuation ring of v_α^*. One containment is clear. To prove the reverse containment, we show that if $f, g \in D[[\{X_\lambda\}]]_3$ and if $v_\alpha^*(f) \geqq v_\alpha^*(g)$, then for some ξ in K, $f/g = \xi f/\xi g$ where $\xi f \in D[[\{X_\lambda\}]]_3$ and $\xi g \in D[[\{X_\lambda\}]]_3 - Q_\alpha$. This is immediate from the approximation theorem for Krull domains [2, P. 12], which shows that there is an element ξ of K such that $v_\alpha(\xi) = -v_\alpha^*(g)$ and such that $v_\beta(\xi) \geqq 0$ for each essential valuation v_β of D distinct from v_α. Hence $(D[[\{X_\lambda\}]])_{Q_\alpha}$ is the valuation ring of v_α^*. Before proving Theorem 3.2, we need to make one final observation: If P_α is finitely generated—say $P_\alpha = (p_1, \cdots, p_n)$—then Q_α is the extension of P_α to $D[[\{X_\lambda\}]]_3$. For is $f \in Q_\alpha$, then $f(s)$ can be written in the form $\sum_{i=1}^n a_i^{(s)} p_i$ for some $a_1^{(s)}, \cdots, a_n^{(s)} \in D$. Hence if f_i is the element of $D[[\{X_\lambda\}]]_3$ such that $f_i(s) = a_i^{(s)}$ for each s in S, then $f = \sum_{i=1}^n f_i p_i$ and f is in the extension of P to $D[[\{X_\lambda\}]]_3$.

Proof of Theorem 3.2. That Q_α is a minimal prime of $D[[\{X_\lambda\}]]_3$ lying over P_α in D is clear. If M is any minimal prime of $D[[\{X_\lambda\}]]_3$ lying over P_α, then our previous observations show that M must be of the form $M_\beta^{(\alpha)}$, since only the $V_\beta^{(\alpha)}$'s meet K in V_α. Hence $V_\beta^{(\alpha)} \supseteqq (D_{P_\alpha}[[\{X_\lambda\}]]_3)_{N_\alpha}$ and $MV_\beta^{(\alpha)}$, the maximal ideal of $V_\beta^{(\alpha)}$, contains $P_\alpha(D_{P_\alpha}[[\{X_\lambda\}]]_3)_{N_\alpha}$. Now $P_\alpha D_{P_\alpha}$ is principal so that $Q_\alpha(D_{P_\alpha}[[\{X_\lambda\}]]_3)_{N_\alpha} = P_\alpha(D_{P_\alpha}[[\{X_\lambda\}]]_3)_{N_\alpha}$. Consequently

$$Q_\alpha \subseteqq Q_\alpha(D_{P_\alpha}[[\{X_\lambda\}]]_3)_{N_\alpha} \cap D[[\{X_\lambda\}]]_3 \subseteqq MV_\beta^{(\alpha)} \cap D[[\{X_\lambda\}]]_3 = M .$$

But since M is a minimal prime of $D[[\{X_\lambda\}]]_3$, this implies that $M = Q_\alpha$ and our proof is complete.

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