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A NOTE ON CERTAIN DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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In this paper an exact solution is obtained for the dual series equations

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x) = f(x)\,, \qquad 0 \leqq x < y\,, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n)} L_n^{(\sigma)}(x) = g(x)\,, \qquad y < x < \infty\,, \tag{2}$$

where $\alpha+\beta+1 > \beta > 1-m$, $\sigma+1 > \alpha+\beta > 0$, m **is a positive integer,**

$$L_n^{(\alpha)}(x) = \binom{\alpha+n}{n} {}_1F_1[-n; \alpha+1; x]\,,$$

is the Laguerre polynomial and $f(x)$ **and** $g(x)$ **are prescribed functions.**

The method used is a generalization of the multiplying factor technique employed by Lowndes [4] to solve a special case of the above equations when

$$\sigma = \alpha,\ A_n = \Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n)C_n,\ \alpha+\beta>0 \quad \text{and} \quad 1>\beta>0\ .$$

In another paper by the present author [5] equations (1) and (2) have been solved by considering separately the equations when (i) $g(x) \equiv 0$, (ii) $f(x) \equiv 0$, and reducing the problem in each case to that of solving an Abel integral equation. Indeed it is easy to verify that the solution obtained earlier [5] is in complete agreement with the one given in this paper.

2. The following results will be required in the analysis.

(i) The orthogonality relation for Laguerre polynomials given by [3, p. 292 (2)] and [3, p. 293 (3)]:

$$\int_0^\infty e^{-x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{mn},\ \alpha > -1\,, \tag{3}$$

where δ_{mn} is the Kronecker delta.

(ii) The formula (27), p. 190 of [2] in the form:

$$\frac{d^m}{dx^m}\{x^{\alpha+m} L_n^{(\alpha+m)}(x)\} = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^\alpha L_n^{(\alpha)}(x)\ . \tag{4}$$

(iii) The following forms of the known integrals [2, p. 191 (30)] and [3, p. 405 (20)]:

$$(5)\qquad \int_0^\xi x^\alpha(\xi - x)^{\beta-1}L_n^{(\alpha)}(x)dx = \frac{\Gamma(\alpha+n+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+n+1)}\xi^{\alpha+\beta}L_n^{(\alpha+\beta)}(\xi)\ ,$$

where $\alpha > -1, \beta > 0$, and

$$(6)\qquad \int_\xi^\infty e^{-x}(x-\xi)^{\beta-1}L_n^{(\alpha)}(x)dx = \Gamma(\beta)e^{-\xi}L_n^{(\alpha-\beta)}(\xi)\ ,$$

where $\alpha + 1 > \beta > 0$.

3. Solution of the equations. Multiplying equation (1) by $x^\alpha(\xi-x)^{\beta+m-2}$, where m is a positive integer, equation (2) by $e^{-x}(x-\xi)^{\sigma-\alpha-\beta}$, and integrating with respect to x over $(0,\xi)$, (ξ, ∞) respectively we find, on using (5) and (6), that

$$(7)\qquad \begin{aligned}&\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n)} L_n^{(\alpha+\beta+m-1)}(\xi)\\ &\quad = \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta+m-1)}\int_0^\xi x^\alpha(\xi-x)^{\beta+m-2}f(x)dx\ ,\end{aligned}$$

where $0 < \xi < y, \alpha > -1, \beta + m > 1$, and

$$(8)\qquad \begin{aligned}&\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n)} L_n^{(\alpha+\beta-1)}(\xi)\\ &\quad = \frac{e^\xi}{\Gamma(\sigma-\alpha-\beta+1)}\int_\xi^\infty e^{-x}(x-\xi)^{\sigma-\alpha-\beta}g(x)dx\ ,\end{aligned}$$

where $y < \xi < \infty, \sigma + 1 > \alpha + \beta > 0$.

If we now multiply equation (7) by $\xi^{\alpha+\beta+m-1}$, differentiate both sides m times with respect to ξ and use the formula (4) we see that it becomes

$$(9)\qquad \begin{aligned}&\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n)} L_n^{(\alpha+\beta-1)}(\xi)\\ &\quad = \frac{\xi^{-\alpha-\beta+1}}{\Gamma(\beta+m-1)}\frac{d^m}{d\xi^m}\int_0^\xi x^\alpha(\xi-x)^{\beta+m-2}f(x)dx\ ,\end{aligned}$$

where $0 < \xi < y, \alpha > -1$, and $\beta + m > 1$.

The left-hand sides of equations (8) and (9) are now identical and an application of the orthogonality relation (3) yields the solution of equations (1) and (2) in the form

$$(10)\qquad \begin{aligned}A_n &= \frac{n!}{\Gamma(\beta+m-1)}\int_0^y e^{-\xi}L_n^{(\alpha+\beta-1)}(\xi)F(\xi)d\xi\\ &\quad + \frac{n!}{\Gamma(\sigma-\alpha-\beta+1)}\int_y^\infty \xi^{\alpha+\beta-1}L_n^{(\alpha+\beta-1)}(\xi)G(\xi)d\xi\ ,\\ &\qquad n = 0, 1, 2, 3, \cdots\ ,\end{aligned}$$

where

$$F(\xi) = \frac{d^m}{d\xi^m}\int_0^\xi x^\alpha(\xi - x)^{\beta+m-2}f(x)dx \tag{11}$$

and

$$G(\xi) = \int_\xi^\infty e^{-x}(x - \xi)^{\sigma-\alpha-\beta}g(x)dx \ , \tag{12}$$

provided that $\alpha + \beta + 1 > 1 - m$ and $\sigma + 1 > \alpha + \beta > 0$, m being a positive integer.

When $\sigma = \alpha$, $A_n = \Gamma(\alpha + n + 1)\Gamma(\alpha + \beta + n)C_n$, the above equations provide the solution to Lowndes' equations for

$$\alpha + \beta > 0, 1 > \beta > 1 - m \ ,$$

and when $m = 1$ the results are in complete agreement (see [4], p. 124). Note also that the dual equations considered recently by Askey [1, p. 683, Th. 3] are essentially the same as Lowndes' equations.

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