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CONCERNING THE INFINITE DIFFERENTIABILITY OF SEMIGROUP MOTIONS

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Let S be a real Banach space. Let C denote the infinitesimal generator of a strongly continuous semigroup T of bounded linear transformations on S. This paper presents a construction which proves that for each $b > 1$ there is a dense subset $D(b)$ of S so that if p is in $D(b)$, then

(A) p is in the domain of C^n for all positive integers n and

(B) $\lim_{n\to\infty} \| C^n p \| (n!)^{-b} = 0$.

Condition (B) will be used in § 3 to obtain series solutions to the partial differential equations $U_{12} = CU$ and $U_{11} = CU$.

Suppose G is a strongly continuous one-parameter group of bounded linear transformations on S which has the property that there is a positive number K so that $|G(x)| < K$ for all numbers x. Let A denote the infinitesimal generator of G. In 1939, Gelfand [1] presented a construction which showed there is a dense subset R of S so that if p is in R, then

(C) p is in the domain of A^n for all positive integers n and

(D) $\lim_{n\to\infty} \| A^n p \| (n!)^{-1} = 0$.

Hille and Phillips, in their work on Semigroups [2], used Gelfand's construction to prove there is a dense subset R of S which satisfies condition (A) with respect to the operator C. Hille and Phillips, however, do not present estimates on the size of $\| C^n p \|$. Also, this author has not been able to use their construction to obtain estimates on the size of $\| C^n p \|$.

2. **Infinite differentiability of semigroup motions.** Let $b > 1$. Let a be a number so that $1 < a < b$. Let M be a positive number so that $| T(X) | < M$ for all nonnegative numbers x less than or equal $\sum_{n=1}^{\infty} n^{-a}$. For each point p in the domain of C (denoted by D_C) and each positive integer n, let $p(n+1, n) = p$. For each point p in D_C and each pair (k, n) of positive integers so that $k \leqq n$, let

$$p(k, n) = k^a \int_0^{k^{-a}} du\, T(u) p(k+1, n) .$$

THEOREM 1. *Suppose p is in D_C and each of k and n is a positive integer. Then*

$$\| p(k, k+n-1) \| \leqq M \| p \| .$$

Proof. Let $w = \prod_{j=0}^{n-1} (k+j)^a$. For each nonnegative integer j, let $r(j) = (k + j)^{-a}$. Then

$$\begin{aligned}&\| p(k, k + n - 1) \| \\ &= w \left\| \int_0^{r(0)} du_0 T(u_0) \int_0^{r(1)} du_1 T(u_1) \cdots \int_0^{r(n-1)} du_{n-1} T(u_{n-1}) p \right\| \\ &= w \left\| \int_0^{r(0)} du_0 \int_0^{r(1)} du_1 \cdots \int_0^{r(n-1)} du_{n-1} T(u_0 + u_1 + \cdots + u_{n-1}) p \right\| < M \| p \| .\end{aligned}$$

THEOREM 2. *Suppose p is in D_C and k is a positive integer. Then*

$$\| p(k, k) - p \| \leqq M \| Cp \| k^{-a} .$$

Proof. Theorem 2 follows from the definition of $p(k, k)$ and the fact that $T(x)p - p = \int_0^x du T(u) Cp$ for all $x > 0$.

THEOREM 3. *Suppose p is in D_C and each of k and n is a positive integer. Then*

$$\| p(k, k + n) - p(k, k + n - 1) \| \leqq M^2 \| Cp \| (k + n)^{-a} .$$

Proof. Let w and $r(j)$ be defined as in the proof of Theorem 1. Then

$$\begin{aligned}&\| p(k, k + n) - p(k, k + n - 1) \| \\ &= (k + n)^a w \left\| \int_0^{r(0)} du_0 T(u_0) \cdots \int_0^{r(n-1)} du_{n-1} T(u_{n-1}) \left[\int_0^{r(n)} du_n (T(u_n)p - p) \right] \right\| \\ &= (k + n)^a w \left\| \int_0^{r(0)} du_0 \cdots \int_0^{r(n-1)} du_{n-1} T(u_0 + \cdots + u_{n-1}) \right. \\ &\qquad \left. \left[\int_0^{r(n)} du_n (T(u_n)p - p) \right] \right\| < M^2 \| Cp \| (k + n)^{-a} .\end{aligned}$$

COROLLARY. *Suppose p is in D_C and k is a positive integer. Then the sequence*

$$S(p, k) \colon p(k, k), p(k, k + 1), p(k, k + 2) ,$$

converges in S.

Proof. Theorem 3 and the fact that $\sum_{n=0}^{\infty} (k + n)^{-a}$ converges imply $S(p, k)$ is a cauchy sequence in S. Since S is complete, $S(p, k)$ will converge.

For each point p in D_C and each positive integer k, let the sequential limit point of $S(p, k)$ be denoted by p_k. Let

$$D(b) \colon \{ p_k \mid p \text{ is in } D_C \text{ and } k \text{ is a positive integer} \} .$$

THEOREM 4. *Suppose p_k is in $D(b)$. Then $p_k \leqq M \| p \|$.*

Proof. Theorem 4 follows from Theorem 1 and the fact that p_k is the sequential limit point of $S(p, k)$.

THEOREM 5. *$D(b)$ is a dense subset of S.*

Proof. Suppose q is in S and q is not in $D(b)$. Let $\varepsilon > 0$. Since D_C is a dense subset of S, there is a point p in D_C so that

(1) $\|p - q\| < \varepsilon/3$.

Theorem 2 implies there is a positive integer k so that

(2) $\|p(k, k) - p\| < \varepsilon/3$ and

(3) $(M + 1)^2 \|Cp\| \sum_{n=0}^{\infty} (k + n)^{-a} < \varepsilon/3$.

Theorem 2, Theorem 3 and statement (3) imply there is a p_k in $D(b)$ so that

(4) $\|p_k - p(k, k)\| < \varepsilon/3$.

Statements (1), (2) and (4) imply $\|p_k - q\| < \varepsilon$. Thus, $D(b)$ is a dense subset of S.

THEOREM 6. *Suppose p_k is in $D(b)$. Then*

$$p_k = k^a \int_0^{k-a} du\, T(u) p_{k+1} \,.$$

Proof. Let $\varepsilon > 0$. Then there is a positive integer n so that

(1) $\|p(k, k + n) - p_k\| < \varepsilon/2$ and

(2) $\|p(k + 1, k + n) - p_{k+1}\| < \varepsilon/2M$.

Statement (2) implies

(3) $\left\| p(k, k + n) - k^a \int_0^{k-a} du\, T(u) p_{k+1} \right\| < \varepsilon/2$.

Theorem 6 now follows from statements (1) and (3).

THEOREM 7. *The elements of $D(b)$ satisfy conditions* (A) *and* (B).

Proof. Suppose p_k is an element of $D(b)$. Theorem 6 implies p_k is in the domain of C^n for all positive integers n and that

(1) $C^n p_k = \prod_{j=0}^{n-1} (k + j)^a \prod_{j=0}^{n-1} [T(1/(k + j)^a) - I] p_{k+n}$.

Thus, the elements of $D(b)$ satisfy condition (A). Statement (1) and Theorem 2 imply

(2) $\|C^n p_k\| \leqq [\prod_{j=0}^{n-1} (k + j)^a](M + 1)^{n+1} \|p\|$.

Statement (2) implies p_k satisfies condition (B). The proof of Theorem 7 is now complete.

3. Partial differential equations in a banach space. The results of § 2 will be used in this section to obtain series solutions to the partial differential equations $U_{12} = CU$ and $U_{11} = CU$. Solutions to these equations may be easily obtained if C is a bounded linear transformation. The transformation C, however, may be unbounded; that is, C may be discontinuous at each point where it is defined.

For each subset D of S, let $P(D)$ denote the set of all functions g for which there is a nonnegative integer n and a sequence $p_0, p_1, \cdots, p_n$ each term of which is in D so that

$$g(x) = \sum_{i=0}^{n} x^i p_i$$

if $x \geqq 0$. If D is a dense subset of S, it may be shown that $P(D)$ is a dense subset of the set of continuous functions from $[0, d] (d > 0)$ to S.

THEOREM 8. *Let $d > 0$. Let b be a number so that $1 < b < 2$. Suppose each of g and h is a function in $P(D(b))$ so that $g(0) = h(0)$. Then there is a function U from $[0, d] \times [0, d]$ to S so that*

(i) *$U_{12}(x, y) = CU(x, y)$ for all (x, y) in $[0, d] \times [0, d]$,*

(ii) *$U(x, 0) = g(x)$ for all x in $[0, d]$ and*

(iii) *$U(0, y) = h(y)$ for all y in $[0, d]$.*

Proof. Suppose n is a nonnegative integer and $p_0, p_1, \cdots, p_n$ is a sequence each term of which is in $D(b)$ so that

$$g(x) = \sum_{i=0}^{n} x^i p_i$$

if $x \geqq 0$. Suppose m is a nonnegative integer and $q_0, q_1, \cdots, q_m$ is a sequence each term of which is in $D(b)$ so that

$$h(y) = \sum_{i=0}^{n} y^i q_i$$

if $y \geqq 0$. Let U be the function from $[0, d] \times [0, d]$ to S so that if (x, y) is in $[0, d] \times [0, d]$, then

$$\begin{aligned}(1)\quad U(x, y) = {} & \textstyle\sum_{i=1}^{n} x^i p_i + \sum_{i=0}^{m} y^i q_i \\ & + \textstyle\sum_{i=1}^{n} \sum_{k=1}^{\infty} (xy)^k x^i C^k p_i/(k!)(i + 1) \cdots (i + k) \\ & + \textstyle\sum_{i=0}^{m} \sum_{k=1}^{\infty} (xy)^k y^i C^k q_i/(k!)(i + 1) \cdots (i + k).\end{aligned}$$

Theorem 7 implies U is well defined on $[0, d] \times [0, d]$. Theorem 7 and the fact that C is a closed transformation imply $U_{12}(x, y) = CU(x, y)$ for all (x, y) in $[0, d] \times [0, d]$. Statement (1) implies $U(x, 0) = g(x)$ and $U(0, y) = h(y)$ for all (x, y) in $[0, d] \times [0, d]$.

THEOREM 9. *Let $d > 0$. Let b be a number so that $1 < b < 2$. Suppose each of g and h is a function in $P(D(b))$. Then there is a function U from $[0, d] \times [0, d]$ to S so that*

(i) *$U_{11}(x, y) = CU(x, y)$ for all (x, y) in $[0, d] \times [0, d]$,*

(ii) *$U(0, y) = g(y)$ if y is in $[0, d]$ and*

(iii) $U_1(0, y) = h(y)$ *if* y *is in* $[0, d]$.

Proof. Let each of g and h be defined as in the proof of Theorem 8. Then let U be the function from $[0, d] \times [0, d]$ to S so that for each (x, y) in $[0, d] \times [0, d]$,

$$(1) \quad U(x, y) = \sum_{i=0}^{n} y^i p_i + x \sum_{i=0}^{m} y^i q_i + \sum_{i=0}^{n} \sum_{k=1}^{\infty} x^{2k} y^i C^k p_i / ((2k)!)(i + 1) \cdots (i + k) + \sum_{i=0}^{m} \sum_{k=1}^{\infty} x^{2k+1} y^i C^k q_i / ((2k + 1)!)(i + 1) \cdots (i + k).$$

An argument analogous to that used in Theorem 8 may be used to show U is well defined on $[0, d] \times [0, d]$ and that U satisfies conditions (i), (ii) and (iii) in the hypothesis of this theorem.

REMARKS. (1) The solution U to the Theorem 8 has the property that for each (x, y) in $[0, d] \times [0, d]$, is in the domain of C^n for all positive integers n. The same remark is true for the solution to the equation in Theorem 9.

(2) Theorem 5 implies there are solutions to $U_{12} = CU$ and $U_{11} = CU$ for a set of boundary functions which is dense in the set of continuous functions from $[0, d]$ to S.

(3) Theorem 9 and Theorem 5 imply there are solutions to the ordinary differential equation $y'' = Cy$ for a dense set of initial values for $y(0)$ and $y'(0)$.

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