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ON GENERAL Z.P.I.-RINGS

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A commutative ring in which each ideal can be expressed as a finite product of prime ideals is called a *general Z.P.I.-ring* (for Zerlegungsatz in Primideale). A general Z.P.I.-ring in which each proper ideal can be uniquely expressed as a finite product of prime ideals is called a *Z.P.I.-ring.* Such rings occupy a central position in multiplicative ideal theory. In case R is a domain with identity, it is clear that R is a Dedekind domain[1] and the ideal theory of R is well known. If R is a domain without identity, the following result of Gilmer gives a somewhat less known characterization of R: If D is an integral domain without identity in which each ideal is a finite product of prime ideals, then each nonzero ideal of D is principal and is a power of D; the converse also holds. Also somewhat less known is the characterization of a general Z.P.I.-ring with identity as a finite direct sum of Dedekind domains and special primary rings.[2]

This paper considers the one remaining case: R is a general Z.P.I.-ring with zero divisors and without identity. A characterization of such rings is given in Theorem 2. This result is already contained ih a more obscure form in a paper by S. Mori. The main contribution here is in the directness of the approach as contrasted to that of Mori.

In order to prove Theorem 2 we need to establish two basic properties of a general Z.P.I.-ring R: R is Noetherian and primary ideals of R are prime powers. Having established these two properties of R, the following result of Butts and Gilmer in [3], which we label as (BG), is applicable and easily yields our characterization of general Z.P.I.-rings without identity.

(BG), [3; Ths. 13 and 14]: *If R is a commutative ring such that $R \neq R^2$ and such that every ideal in R is an intersection of a finite number of prime power ideals, then $R = F_1 \oplus \cdots \oplus F_k \oplus T$ where each F_i is a field and T is a nonzero ring without identity in which every nonzero ideal is a power of T.*

It is important to note that we do not use Butts and Gilmer's

[1] M. Sono [**14**] and E. Noether [**13**] were among the first to consider Dedekind domains. For a historical development of the theory of Dedekind domains see [**4**; pp. 31-32].

[2] S. Mori in [**11**] considered both general Z.P.I.-rings with identity and Z.P.I.-rings without identity which contain no proper zero divisors, but Mori's results in these cases are not as sharp as those of Asano and Gilmer.

paper [3] to prove that a general Z.P.I.-ring is Noetherian, while Butts and Gilmer do use this result from Mori's paper [11; Th. 7]. Theorem 2 gives a finite direct sum characterization of a general Z.P.I.-ring whereas Theorems 3 and 4 and Corollary 2 give characterizations of a general Z.P.I.-ring in terms of ideal-theoretic conditions.

Since we are only concerned with commutative rings, "ring" will always mean "commutative ring". The notation and terminology is that of [16] with two exceptions: $\subseteqq$ denotes containment and $\subset$ denotes proper containment, and we do not assume that a Noetherian ring contains an identity. If A is an ideal of a ring R, we say that A is a *prper ideal* of R if $(0) \subset A \subset R$ and that A is a *genuine ideal* of R if $A \subset R$.

2. Structure theorem of a general Z.P.I.-ring. In this section section we prove directly that a general Z.P.I.-ring is Noetherian by proving that each of its prime ideals is finitely generated. We then use result (BG) to prove the structure theorem of a general Z.P.I.-ring.

DEFINITION. Let R be a ring. If there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of $n+1$ prime ideals of R where $P_n \subset R$, but no such chain of $n+2$ prime ideals, then we say that R *has dimension* n and we write $\dim R = n$.

LEMMA 1. *If R is a general* Z.P.I.-*ring, R contains only finitely many minimal prime ideals and* $\dim R \leqq 1$.

Proof. If R contains no proper prime ideal, then the lemma is clearly true. Therefore, we assume R contains a proper prime ideal P and we show that R contains a minimal prime ideal. If P is not a minimal prime of R, there exists a prime ideal P_1 such that $P_1 \subset P \subset R$. It follows that R/P_1 is a domain containing a proper prime ideal in which each ideal can be represented as the product of finitely many prime ideals. This implies that R/P_1 is a Dedekind domain [6]. Therefore, P_1 is a minimal prime of R. This also shows that $\dim R \leqq 1$.

Since R is a general Z.P.I.-ring, there exist prime ideals $Q_1, \cdots, Q_n$ in R and positive integers $e_1, \cdots, e_n$ such that $(0) = Q_1^{e_1} \cdots Q_n^{e_n}$. If M is a minimal prime ideal of R, $(0) = Q_1^{e_1} \cdots Q_n^{e_n} \subseteqq M$ which implies that $Q_i \subseteqq M$ for some i. Hence, $M = Q_i$ and it follows that the collection $\{Q_1, \cdots, Q_n\}$ contains all the minimal prime ideals of R. Therefore, R contains only finitely many minimal prime ideals.

LEMMA 2. *If R is a general* Z.P.I.-*ring containing a genuine* *prime ideal, then each minimal prime ideal of R is finitely generated.*

Proof.[3] Let P be a minimal prime ideal of R and let $\{P_1, \cdots, P_n\}$ be the collection of minimal primes of R distinct from P. If $P = (0)$, the proof is clear. If $(0) \subset P$, we show that P is finitely generated by an inductive argument; that is, we show how to select a finite number of elements in P which generate P. We divide the proof into three cases.

Case 1. $P = P^2$. Since $P = P^2 \subseteqq RP \subseteqq P$, $P = RP$. Now,

$$P \nsubseteqq \bigcup_{i=1}^n P_i$$

since $P \nsubseteqq P_i$ for $1 \leqq i \leqq n$ so let $x_1 \in P \setminus (\bigcup_{i=1}^n P_i)$. Thus, there exist prime ideals $M_1, \cdots, M_s$, positive integers $e_0, e_1, \cdots, e_s$, and a non-negative integer e_{s+1} such that

$$(x_1) = P^{e_0} M_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} = P M_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} = P M_1^{e_1} \cdots M_s^{e_s}$$

since $P = RP$. Let $\delta = \sum_{i=1}^s e_i$. If $P = (x_1)$, we are done. If $(x_1) \subset P$, then by choice of x_1 each M_i is a maximal prime ideal of R. Then [2; Proposition 2, p. 70] implies that $P \nsubseteqq \{(x_1) \cup (\bigcup_{i=1}^n P_i)\}$. If $x_2 \in P \setminus \{(x_1) \cup (\bigcup_{i=1}^n P_i)\}$, then

$$(x_2) = P M_1^{f_1} \cdots M_s^{f_s} R^{f_{s+1}} Q_1^{g_1} \cdots Q_t^{g_t} = P M_1^{f_1} \cdots M_s^{f_s} Q_1^{g_1} \cdots Q_t^{g_t}$$

where Q_j is a maximal prime ideal of R for $1 \leqq j \leqq t$, $f_i \in \omega_0$ for $1 \leqq i \leqq s+1$, and $g_j \in w$ for $1 \leqq j \leqq t$. Since $(x_2) \nsubseteqq (x_1)$, we have that $e_{i_0} > f_{i_0}$ for some i_0, $1 \leqq i_0 \leqq s$. Therefore,

$$\begin{aligned}(x_1, x_2) &= P M_1^{e_1} \cdots M_s^{e_s} + P M_1^{f_1} \cdots M_s^{f_s} Q_1^{g_1} \cdots Q_t^{g_t} \\ &= P M_1^{m_1} \cdots M_s^{m_s} (M_1^{e_1 - m_1} \cdots M_s^{e_s - m_s} + M_1^{f_1 - m_1} \cdots M_s^{f_s - m_s} Q_1^{g_1} \cdots Q_t^{g_t})\end{aligned}$$

where $m_i = \min\{e_i, f_i\}$ for $1 \leqq i \leqq s$. By the definition of m_i, if $e_i - m_i \neq 0$, then $f_i - m_i = 0$, and if $f_i - m_i \neq 0$, then $e_i - m_i = 0$. Let $A = M_1^{e_1 - m_1} \cdots M_s^{e_s - m_s}$ and let $B = M_1^{f_1 - m_1} \cdots M_s^{f_s - m_s} Q_1^{g_1} \cdots Q_t^{g_t}$, we show that $A + B$ is contained in no maximal prime ideal of R. Note that $e_{i_0} - m_{i_0} \neq 0$ since $e_{i_0} > f_{i_0}$. If M is a maximal prime ideal of R containing A, then there exists a k, $1 \leqq k \leqq s$, such that $e_k - m_k \neq 0$ and $M_k \subseteqq M$. Since M_k is a maximal prime ideal of R, it follows that $M = M_k$. Now, $e_k - m_k \neq 0$ implies that $f_k - m_k = 0$ which shows that $B \nsubseteqq M_k = M$. Thus, if M is a maximal prime ideal of R containing A, M does not contain B. It follows that $A + B$ is contained in no maximal prime ideal of R. Therefore, there exists a positive integer λ such that $A + B = R^\lambda$ and $(x_1, x_2) = P M_1^{m_1} \cdots M_s^{m_s}(A + B) = P M_1^{m_1} \cdots M_s^{m_s} R^\lambda = P M_1^{m_1} \cdots M_s^{m_s}$. By our choice of m_i, we have $e_i \geqq m_i$ for $1 \leqq i \leqq s$. But $e_{i_0} < f_{i_0} = m_{i_0}$ implies that

[3] The proof of Lemma 2 was suggested to the author by Professor Gilmer.

$\delta - 1 \geqq \sum_{i=1}^{s} m_i \geqq 0$.

Assume that we have chosen, as described above, $x_1, x_2, \cdots, x_u$ in P such that $(x_1, \cdots, x_u) = PM_1^{v_1} \cdots M_s^{v_s}$ and $\delta - (u-1) \geqq \sum_{i=1}^{s} v_i \geqq 0$. Then by the above method, either $P = (x_1, \cdots, x_u)$ or there exists an $x_{u+1} \in P \diagdown \{(x_1, \cdots, x_u) \cup (\bigcup_{i=1}^{n} P_i)\}$ such that

$$(x_1, \cdots, x_u, x_{u+1}) = PM_1^{v_1'} \cdots M_s^{v_s'}$$

where $v_i' \in \omega_0$ and $\delta - (u + 1 - 1) \geqq \sum_{i=1}^{s} v_i' \geqq 0$. Since $\sum_{i=1}^{s} e_i$ is a finite positive number, there exists a positive integer q and $x_1, \cdots, x_q \in P$ such that $P = (x_1, \cdots, x_q)$; that is, P is a finitely generated ideal of R.

Case 2. $P^2 \subset P$ and $P = RP$. Now, $P \nsubseteq \{P^2 \cup (\bigcup_{i=1}^{n} P_i)\}$ by [2; Proposition 2, p. 70] so let $x_1 \in P \diagdown \{P^2 \cup (\bigcup_{i=1}^{n} P_i)\}$. Then there exist prime ideals $M_1, \cdots, M_s$ of R, $e_1, \cdots, e_s \in \omega$, and $e_{s+1} \in \omega_0$ such that $(x_1) = PM_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} = PM_1^{e_1} \cdots M_s^{e_s}$ since $P = RP$. If $P = (x_1)$ we are done. If $(x_1) \subset P$, then we can choose an

$$x_2 \in P \diagdown \{(x_1) \cup P^2 \cup (\bigcup_{i=1}^{n} P_i)\}$$

by [2; Proposition 2, p. 70]. We now consider (x_1, x_2) and the remainder of the proof of Case 2 is the same as the proof of Case 1. Thus, P is a finitely generated ideal of R.

Case 3. $P^2 \subset P$ and $RP \subset P$. Let $x \in P \diagdown RP$. Then there exist prime ideals $M_1, \cdots, M_s$ of R and $e_1, \cdots, e_{s+1} \in \omega_0$ such that $(x) = PM_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} \nsubseteq RP$. Thus, $e_i = 0$ for $1 \leqq i \leqq s + 1$; that is, $P = (x)$.

LEMMA 3. *Each prime ideal of a general* Z.P.I.*-ring is finitely generated.*

Proof. Let R be a general Z.P.I.-ring.

Case 1. R contains no proper prime ideal. If $R = R^2$, let $r \in R \diagdown \{0\}$. Since R is a general Z.P.I.-ring, there exists a positive integer n such that $(r) = R^n = R$. If $R^2 \subset R$, let $r \in R \diagdown R^2$. Then $(r) = R$.

Case 2. R contains a proper prime ideal. Let M be a nonzero prime ideal of R. If M is a minimal prime ideal of R, M is finitely generated by Lemma 2. If M is not a minimal prime ideal of R, the proof of Lemma 1 implies that there exists a minimal prime ideal P of R such that $P \subset M$. Thus, R/P is Noetherian which implies that M/P is a finitely generated ideal of R/P. Since P is a finitely generated ideal of R, it follows that M is a finitely generated ideal of R.

Thus, each prime ideal of R is finitely generated.

THEOREM 1. *A general* Z.P.I.*-ring is Noetherian.*

Proof. Let A be an ideal of R, a general Z.P.I.-ring. Then there exist prime ideals $P_1, \cdots, P_n$ of R and positive integers $e_1, \cdots, e_n$ such that $A = P_1^{e_1} \cdots P_n^{e_n}$. Since each P_i is finitely generated by Lemma 3, it follows that A is finitely generated. Thus, R is Noetherian.

REMARK. Theorem 1 also follows from the fact that a ring R is Noetherian if and only if each prime ideal of R is finitely generated. [4; Th. 2].

RESULT 1. *If Q is a P-primary ideal in a ring R such that Q can be represented as a finite product of prime ideals, then Q is a power of P.*

Proof. By hypothesis there exist distinct prime ideals $P_1, \cdots, P_n$ and positive integers $e_1, \cdots, e_n$ such that $Q = P_1^{e_1} \cdots P_n^{e_n}$. Since $Q = P_1^{e_1} \cdots P_n^{e_n} \subseteqq P$, $P_i \subset P$ for some i—say $i = 1$. Now, $P = \sqrt{Q} = P_1 \cap \cdots \cap P_n$ which implies that $P \subseteqq P_i$ for each i. Therefore, $P \subseteqq P_1 \subseteqq P$; that is, $P_1 = P$. We have that $Q = P^{e_1} P_2^{e_2} \cdots P_n^{e_n}$ where $P \subset P_i$ for $2 \leqq i \leqq n$. Since

$$Q = P^{e_1}(P_2^{e_2} \cdots P_n^{e_n}) \subseteqq Q$$

and $P_2^{e_2} \cdots P_n^{e_n} \not\subseteqq P$, it follows that $P^{e_1} \subseteqq Q$. Hence, $Q = P^{e_1}$.

DEFINITIONS. Let R be a ring. We say that R *has property* (α), if each primary ideal of R is a power of its (prime) radical [3]. If each ideal of R is an intersection of a finite number of prime power ideals, we say that R *has property* (δ) [3]. Finally, we say that R *satisfies property* $(\sharp)$ if R is a ring without identity such that each nonzero ideal of R is a power of R.

REMARK. If R is a ring satisfying property $(\sharp)$, it follows that either R is an integral domain in which $\{R^i\}_{i=1}^{\infty}$ is the collection of nonzero ideals of R or R is not an integral domain and $\{R, R^2, \cdots, R^n = (0)\}$ is the collection of all ideals of R for some $n \in \omega$.

COROLLARY 1. *A general* Z.P.I.*-ring has property* (α).

Proof. This follows immediately from Result 1.

THEOREM 2. *Structure theorem of a general* Z.P.I.*-ring. A ring R is a general* Z.P.I.*-ring if and only if R has the following structure*:

(a) *If* $R = R^2$, *then* $R = R_1 \oplus \cdots \oplus R_n$ *where* R_i *is either a Dedekind domain or a special* P.I.R. *for* $1 \leqq i \leqq n$.

(b) *If* $R \neq R^2$, *then either* $R = F \oplus T$ *or* $R = T$ *where* F *is a field and* T *is a ring satisfying property* (♯).

Proof. ($\rightarrow$) If R is a general Z.P.I.-ring, then R is Noetherian and has property (α). Hence, [3; Corollary 6] implies that (δ) holds in R. If $R = R^2$, then R contains an identity by [5; Corollary 2]. Therefore, [1; Th. 1] implies that part (a) holds. If $R \neq R^2$, then by (BG) $R = F_1 \oplus \cdots \oplus F_u \oplus T$ where each F_i is a field and T is a nonzero ring satisfying property (♯). Using a contrapositive argument, we show that $u \ngeqq 2$.

Assume that $u \geqq 2$. We show that R is not a general Z.P.I.-ring. Since $u \geqq 2$, it is clear that T is an ideal of R that is not prime. The prime ideals of R containing T are R and

$$P_i = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_u \oplus T$$

for $1 \leqq i \leqq u$ where $T \subset P_i$ for each i. Now

$$P_iP_j = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_{j-1} \oplus (0) \oplus F_{j+1} \oplus \cdots \oplus F_u \oplus T^2 ,$$

$$RP_i = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_u \oplus T^2 ,$$

and $R^2 = F_1 \oplus \cdots \oplus F_u \oplus T^2$. Since $T^2 \subset T$, it follows that $T \nsubseteqq P_iP_j$, $T \nsubseteqq RP_i$, and $T \nsubseteqq R^2$ for $1 \leqq i, j \leqq u$. Thus, T cannot be represented as a finite product of prime ideals of R; that is, R is not a general Z.P.I.-ring. Therefore, if R is a general Z.P.I.-ring, $u \ngeqq 2$; that is, $R = F_1 \oplus T$ or $R = T$ where F_1 is a field and T is a ring satisfying property (♯).

($\leftarrow$) If R is a direct sum of finitely many Dedekind domains and special P.I.R.'s R is a general Z.P.I.-ring by [1; Th. 1]. If $R = T$ where T is a ring satisfying property (♯), then R is clearly a general Z.P.I.-ring. If $R = F \oplus T$ where F is a field and T is a ring satisfying property (♯), then $\{F \oplus T^i, T^i, (0): i \in \omega\}$ is the collection of ideals of R. It follows that each ideal of R is a finite product of prime ideals. Therefore, if R satisfies either (a) or (b), R is a general Z.P.I.-ring.

3. Necessary and sufficient conditions on a general Z.I.P.-ring.

In this section we again use results of Butts and Gilmer in [3] to derive several necessary and sufficient conditions for a ring to be a general Z.P.I.-ring.

DEFINITION. Let A be an ideal of a ring R. We say that A

is simple if there exist no ideals properly between A and A^2. To avoid conflicts with other definitions of a simple ring we say in case $A = R$ that R *satisfies property S*.

LEMMA 4. *Let A be an ideal of a Noetherian ring R. If $B = \bigcap_{i=1}^{\infty} A^i$, then $AB = B$.*

Proof. See [15; L_1].

LEMMA 5. *If A is a genuine ideal of a Noetherian domain D, then $\bigcap_{i=1}^{\infty} A^i = (0)$.*

Proof. Let K be the quotient field of D and let $D^* = D[e]$ where e is the identity of K. Then D^* is Noetherian by [5; Th. 1], and since A is also an ideal of D^*, [16; Corollary 1, *p.* 216] shows that $\bigcap_{i=1}^{\infty} A^i = (0)$.

LEMMA 6. *Let A be a simple ideal of a ring R. Then for each $i \in \omega$ there are no ideals properly between A^i and A^{i+1}. Further, the only ideals between A and A^n for $n \in \omega$ are $A, A^2, \cdots, A^n$.*

Proof. See [7; Lemma 3].

LEMMA 7. *Let A be a proper simple ideal of a Noetherian ring R. If there exists a prime ideal P of R such that $(0) \subset P \subset A \subset R$, P is unique and $P = \bigcap_{i=1}^{\infty} A^i$. Also, if Q is a P-primary ideal of R, $Q = P$.*

Proof. We first show by an inductive argument that $P \subset A^i$ for each $i \in \omega$. By hypothesis $P \subset A$. Assume that $P \subset A^k$ for some $k \in \omega$. Since A/P is a proper ideal of R/P, a Noetherian integral domain, $A^k/P \supset (A^k/P)(A/P) = (A^{k+1} + P)/P \supset P/P$ by [5; Corollary 1] which shows that $A^k \supset A^{k+1} + P \supseteqq A^{k+1}$. Therefore, $A^{k+1} + P = A^{k+1}$. Since $A^{k+1} + P \supset P$, it follows that $P \subset A^{k+1}$. Thus, $P \subset A^i$ for each $i \in \omega$.

We now show that $P = \bigcap_{i=1}^{\infty} A^i$. Since A/P is a proper ideal of a Noetherian domain, $P/P = \bigcap_{i=1}^{\infty}(A/P)^i$ by Lemma 5. Also, since $\bigcap_{i=1}^{\infty}(A/P)^i = \bigcap_{i=1}^{\infty}((A^i + P)/P) = \bigcap_{i=1}^{\infty}(A^i/P) = (\bigcap_{i=1}^{\infty} A^i)/P$, it follows that $P = \bigcap_{i=1}^{\infty} A^i$.

Finally, we show that if Q is a P-primary ideal of R, then $Q = P$. Lemma 4 shows that $P = A(\bigcap_{i=1}^{\infty} A^i) = AP$. There exists an $a \in A$ such that $ap = p$ for each $p \in P$ by [5; Corollary 1]; that is, $ap - p = 0$ for each $p \in P$. If $x \in R \setminus A$, then $p(ax - x) = apx - px = 0 \in Q$ for each $p \in P$. Since $x \notin A$, $ax - x \notin A$ which shows that $ax - x \notin P$. Thus, $p \in Q$ for each $p \in P$ since $p(ax - x) \in Q$ for each

$p \in P$, $ax - x \notin P$, and Q is a P-primary ideal of R. Thus, $P \subseteqq Q$ which shows that $Q = P$.

THEOREM 3. *Let R be a ring.*

(A) *If R contains an identity, then R is a general* Z.P.I.*-ring if and only if R satisfies the following two conditions*:

(1) *R is Noetherian.*

(2) *Each maximal ideal of R is simple.*

(B) *If R does not contain an identity and R contains a proper prime ideal, then R is a general* Z.P.I.*-ring if and only if R satisfies the following four conditions*:

(1) *R is Noetherian.*

(2) *R satisfies property S.*

(3) *Each maximal prime ideal of R is simple.*

(4) *$\bigcap_{i=1}^{\infty} R^i$ is a field.*

(C) *If R does not contain an identity and R contains no proper prime ideal, then R is a general* Z.P.I.*-ring if and only if R satisfies the following two conditions*:

(1) *R is Noetherian.*

(2) *R satisfies property S.*

Proof of (A). Part (A) follows immediately from [1; Th. 5].

Proof of (B). ($\rightarrow$) Assume that R is a general Z.P.I.-ring. Then R is Noetherian by Theorem 1. Since R contains a proper prime ideal, Theorem 2 shows that $R = F \oplus T$ where F is a field and T is a ring satisfying property ($\sharp$). Hence, R clearly satisfies property S. If T is a domain, then F and T are the maximal prime ideals of R. If T is not a domain, then T is the maximal prime ideal of R. It follows that each maximal prime ideal of R is simple. Finally, $\bigcap_{i=1}^{\infty} R^i = \bigcap_{i=1}^{\infty}(F \oplus T)^i = F$, a field.

($\leftarrow$) Assume that conditions (1), (2), (3), and (4) hold. Let Q be a P-primary ideal of R. If $P = R$ or if P is a maximal prime ideal of R, there exists an integer n such that $P^n \subseteqq Q$ since R is Noetherian. Hence Lemma 6 shows that there exists an integer k such that $Q = P^k$. If P is a proper nonmaximal prime ideal of R, there exists a maximal prime ideal M of R such that $P \subset M \subset R$, and it follows from Lemma 7 that $Q = P$. Thus, R is a Noetherian ring having property (α) which shows that (δ) holds in R. [3; Corollary 6]. Therefore, by (BG) $R = F_1 \oplus \cdots \oplus F_m \oplus T$ where each F_i is a field and T satisfies property ($\sharp$). Since R contains a proper prime ideal, $m \geqq 1$; condition (4) implies that $m \not> 1$. Hence $R = F_1 \oplus T$ which implies that R is a general Z.P.I.-ring.

Proof of (C). ($\rightarrow$) If R is a general Z.P.I.-ring containing no proper prime ideal, then $R = T$ where T is a ring satisfying property ($\sharp$), Hence, R is Noetherian and satisfies property S.

($\leftarrow$) Assume that conditions (1) and (2) hold. Since R is Noetherian and since R is the only nonzero prime ideal in R, R has property (α). Thus, R is a general Z.P.I.-ring by an argument similar to that given in part (B) above.

LEMMA 8. *A ring R has property (δ) if and only if R satisfies the following three conditions*:

(1) *R is Noetherian.*

(2) *R satisfies property S.*

(3) *Each maximal prime ideal of R is simple.*

Proof. ($\rightarrow$) Assume that R has property (δ). If $R = R^2$, [3; Th. 11] implies that R is a general Z.P.I.-ring. Therefore, (1), (2), and (3) hold by Theorem 3. If $R \neq R^2$, then [3; Th. 12] implies that R is Noetherian. From (BG) we have that $R = F_1 \oplus \cdots \oplus F_m \oplus T$ where each F_i is a field and T satisfies property ($\sharp$). It follows from the representation of R, that (2) and (3) hold.

($\leftarrow$) We showed in the proof of Theorem 3 (B) that if (1), (2), and (3) hold in a ring R, then (δ) holds in R.

LEMMA 9. *In a Noetherian ring R, property (α) is equivalent to the following two conditions*:

(2) *R satisfies property S.*

(3) *Each maximal prime ideal of R is simple.*

Proof. This follows immediately from Lemma 8 and [3; Corollary 6].

THEOREM 4. *If R is a ring with identity, R is a general* Z.P.I.-*ring if and only if R is Noetherian and (α) holds in R.*

Proof. The necessity follows from Theorem 1 and Corollary 1 and the sufficiency follows from [3; Corollary 6 and Th. 11].

COROLLARY 2. *Let R be a ring without identity.*

(A) *If R contains a proper prime ideal, then R is a general* Z.P.I.-*ring if and only if R satisfies the following three conditions*:

(1) *R is Noetherian.*

(2′) (α) *holds in R.*

(4) $\bigcap_{i=1}^{\infty} R^i$ *is a field.*

(B) *If R contains no proper prime ideal, then R is a general*

Z.P.I.-ring if and only if R satisfies the following two conditions:
(1) *R is Noetherian.*
(2′) *(α) holds in R.*

Proof. This follows immediately from Theorem 3 and Lemma 9.

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