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SOME RENEWAL THEOREMS CONCERNING A SEQUENCE OF CORRELATED RANDOM VARIABLES

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Consider a sequence $\{x_n\}, n = 1, 2, \cdots$ of random variables. Let $F_n(x)$ be the distribution function of $S_n = \sum_{k=1}^{n} x_k$ and $H_n(x)$, the distribution function of $M_n = \max_{1 \leqq k \leqq n} S_k$. Here we study the asymptotic behaviour of

1.1 $$\sum_{n=1}^{\infty} a_n G_n(x) ,$$

where $G_n(x)$ is to mean either $F_n(x)$ or $H_n(x)$ (so that if a property holds for both $F_n(x)$ and $H_n(x)$ it holds for $G_n(x)$ and conversely) and $\{a_n\}$ a suitable positive term sequence, when $\{x_n\}$ form

(i) a sequence of dependent random variables such that the correlation between x_i and x_j is $\rho, i \neq j, i, j = 1, 2, \cdots$, $0 < \rho < 1$, $E(x_i) = \mu_i$, $i = 1, 2, \cdots$ and

1.2 $$\lim_{n \to \infty} \frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n^\alpha} = \mu, \alpha > 1, 0 < \mu < \infty$$

and

(ii) a sequence of identically distributed random variables with $E(x_i) = \mu, i = 1, 2, \cdots$ such that the correlation between x_i and x_j is $\rho_{ij} = \rho^{|i-j|}, i, j = 1,2, \cdots, 0 < \rho < 1$.

Suitable examples are worked out to illustrate the general theory.

Let $N(x)$ be the first value of n such that $S_n \geqq x$, $x > 0$. $N(x)$ is a random variable and let

1.3 $$H(x) = E\{N(x)\} .$$

$H(x)$ is called the renewal function and much research work has been done with reference to the study of the asymptotic behaviour of $H(x)$ as $x \to \infty$. Feller has shown that

1.4 $$\lim_{x \to \infty} H(x)/x = 1/\mu ,$$

when $\{x_n\}$ form a sequence of independent and identically distributed random variables with $\mu = E(x_n), 0 < \mu < \infty$, the limit being interpreted as zero when $\mu = \infty$. Blackwell has generalised the above, by considering the renewal process $N(x, h)$ which denotes the number of renewals occuring in the interval $(x, x + h]$. He has shown that, for any fixed $h, (h > 0)$, if

1.5 $$H(x, h) = E\{N(x, h)\} ,$$

then

1.6 $$\lim_{x\to\infty} H(x, h) = h/\mu .$$

This has been proved earlier by Doob for the discrete case. Tatsuo Kawata has extended this further. He has proved that

1.7 $$\lim_{x\to\infty} \sum_{n=1}^{\infty} a_n P(x < S_n \leqq x + h) = ha/\mu ,$$

where

1.8 $$(1/n) \sum_{k=1}^{n} a_k = a + o(1/\sqrt{n}) .$$

He has also shown that if 1.8 is replaced by

1.9 $$(1/n) \sum_{k=1}^{n} a_k = a + o(1/n^{\alpha}) , \qquad \alpha < 1/2 ,$$

then 1.7 does not hold.

Herbert Robbins and Y.S. Chow have relaxed the restriction of independence and obtained a renewal theorem for the dependent case. They have shown that if

1.10 $$E(x_n \mid x_1, x_2, \cdots, x_{n-1}) = E(x_n) = \mu_n(\text{constant}) ,$$

1.11 $$\lim_{n\to\infty} \frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n} = \mu , \qquad 0 < \mu < \infty$$

and for some $\alpha > 1$

1.12 $$E\{\mid x_n - \mu_n \mid^{\alpha} \mid x_1, x_2, \cdots, x_{n-1}\} \leqq k < \infty ,$$

then

1.13 $$\lim_{x\to\infty} H(x)/x = 1/\mu .$$

C.C. Heyde has proved that if $\{x_n\}$ is a sequence of independent and identically distributed random variables with mean μ, $0 < \mu < \infty$, then

1.14 $$\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{\alpha L(x)}{\Gamma(1 + r)} (x/\mu)^r , \qquad x \to \infty ,$$

where a_n's are positive term coefficient sequences such that

1.15 $$\sum_{n=1}^{\infty} a_n x^n \sim \frac{\alpha L[(1 - x)^{-1}]}{(1 - x)^r} , \qquad x \to 1^- ,$$

α, r are real numbers greater than zero and $L(x)$ is some nonnegative function of slow growth.

Here we extend the above theorem to the two cases (i) and (ii) given in the beginning. Subject to suitable restrictions we have shown that in the first case

1.16
$$\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)}{(\lambda+1)} , \qquad x \to \infty$$

and in the second case

1.17
$$\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{(x/\mu)^{\lambda+1} L(x)}{(\lambda+1)} , \qquad x \to \infty$$

where

1.18
$$a_n \sim n^\lambda L(n), \qquad n \to \infty$$

λ being chosen such that $\sum_{n=1}^{\infty} a_n$ is divergent.

We illustrate 1.16 for the particular case when $\{x_i\}$ follow the normal law with mean μ_i and variance one and 1.17 for the cases when they follow (i) the normal law with mean μ and variance one and (ii) the type III distribution with density function

1.19
$$\begin{aligned} f(x) &= [\Gamma(r)]^{-1}\theta^{-r} e^{-x/\theta} x^{r-1} , \qquad && x \geqq 0 , \\ &= 0 , && x < 0 . \end{aligned}$$

For the type III distribution we also prove that

1.20
$$\sum_{n=1}^{\infty} a_n P(x < S_n \leqq x + h) \sim (h/r\theta)(x/r\theta)^\lambda L(x) , \qquad x \to \infty .$$

2. A lemma. We use the following lemma extensively.

LEMMA 2.1. *Let $L(x)$ be such that $L(cx) \sim L(x)$ for every positive c as x tends to infinity. If*

2.11
$$a_n \sim n^\lambda L(n) , \qquad n \to \infty ,$$

λ being chosen such that $\sum a_n$ is divergent, then

2.12
$$\sum_{n=1}^{\infty} a_n e^{-n^\theta s} \sim (1/\theta)\Gamma[(\lambda+1)/\theta] s^{-(\lambda+1)/\theta} L(1/s^\theta) , \qquad s \to 0, \theta > 0 ,$$

2.13
$$\sum_{n=1}^{\infty} a_n n^\theta e^{-n^\theta s} \sim (1/\theta)\Gamma[(\lambda+\theta+1)/\theta] s^{-(\lambda+\theta+1)/\theta} L(1/s^\theta) ,$$
$$s \to 0, \theta > 0 ,$$

2.14
$$\sum_{n=1}^{\infty} a_n e^{-nms} \sim \Gamma(\lambda+1)(sm)^{-(\lambda+1)} L(1/s) , \qquad s \to 0 ,$$

2.15 $$\sum_{n=1}^{\infty} na_n e^{-nms} \sim \Gamma(\lambda+2)(sm)^{-(\lambda+2)}L(1/s)\ , \qquad s \to 0\ ,$$

These can be got from Corollary 1(a) of [8, p. 182] by proper substitutions.

3. Renewal theorems.

THEOREM 3.1. *Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of dependent random variables such that the correlation between any two variables x_i and x_j is ρ, $i \neq j$, $i, j = 1, 2, \cdots$ and $0 < \rho < 1$. Let $E(x_i) = \mu_i$, $i = 1, 2, \cdots$. If*

3.1.1 $$\lim_{n\to\infty} \frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n^\alpha} = \mu\ , \qquad \alpha > 1, 0 < \mu < \infty\ ,$$

3.1.2 $$1 - H_n(n^\alpha x) \leqq p(n, x)\ ,$$

where $p(n, x)$ satisfies

3.1.3 $$\delta_n = \int_\mu^\infty p(n, x)dx \to 0\ , \qquad n \to \infty\ ,$$

the nonnegative constants a_n satisfy 2.11 *and the condition*

3.1.4 $$\sum_{n=1}^{\infty} a_n F_n(n^\alpha \beta) < \infty\ , \qquad 0 < \beta < \mu\ ,$$

then

3.1.5 $$\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)}{(\lambda+1)}\ , \qquad x \to \infty\ .$$

Proof of Theorem 3.1. *Let*

3.1.6 $$\begin{aligned}\phi(x) &= \sum_{n=1}^{\infty} a_n G_n(x) U(x - n^\alpha \beta)\ . \\ &= \sum_{n=1}^{\infty} a_n U(x - n^\alpha \mu) - \sum_{n=1}^{\infty} a_n [U(x - n^\alpha \mu) - G_n(x)] U(x - n^\alpha \beta)\ ,\end{aligned}$$

where

3.1.7 $$\begin{aligned}U(x) &= 1\ , \qquad x \geqq 0\ , \\ &= 0\ , \qquad x < 0\ .\end{aligned}$$

Let

3.1.8 $$\phi(s) = \int_0^\infty e^{-sx}\phi(x)dx\ .$$

Then we have

3.1.9 $$\phi(s) = s^{-1}\sum_{n=1}^{\infty} a_n e^{-n^{\alpha}\mu s} - \sum_{n=1}^{\infty} a_n(L_n - K_n) ,$$

where

3.1.10 $$L_n = \int_{n^{\alpha}\mu}^{\infty} e^{-sx}[1 - G_n(x)]dx, \qquad K_n = \int_{\beta n^{\alpha}}^{\mu n^{\alpha}} e^{-sx}G_n(x)dx ,$$

the term by term integration is justified by the monotone convergence. Now using 2.12, we have

3.1.11 $$s^{-1}\sum_{n=1}^{\infty} a_n e^{-n^{\alpha}\mu s} \sim \frac{\Gamma[(\lambda + 1)/\alpha]s^{-\{[(\lambda+1)/\alpha]+1\}}L(1/s^{\alpha})}{\alpha\mu^{(\lambda+1)/\alpha}} .$$

Also

3.1.12 $$\begin{aligned} L_n &= \int_{n^{\alpha}\mu}^{\infty} e^{-sx}[1 - G_n(x)]dx \\ &= n^{\alpha}\int_{\mu}^{\infty} e^{-n^{\alpha}sx}[1 - G_n(n^{\alpha}x)]dx \\ &\leqq n^{\alpha}e^{-n^{\alpha}\beta s}\int_{\mu}^{\infty}[1 - G_n(n^{\alpha}x)]dx . \end{aligned}$$

Using 3.1.3 and the fact that $G_n(x) \leqq F_n(x)$, we get

3.1.13 $$\int_{\mu}^{\infty}[1 - G_n(n^{\alpha}x)]dx \to 0 , \qquad n \to \infty .$$

Hence we may write

3.1.14 $$L_n = n^{\alpha}e^{-n^{\alpha}\beta s}\delta_n ,$$

where $\delta_n \to 0$ as $n \to \infty$ uniformly in $s > 0$.

3.1.15 $$\begin{aligned} K_n &= \int_{\beta n^{\alpha}}^{\mu n^{\alpha}} e^{-sx}G_n(x)dx \\ &= n^{\alpha}\int_{\beta}^{\mu} e^{-n^{\alpha}sx}G_n(n^{\alpha}x)dx \\ &\leqq n^{\alpha}e^{-n^{\alpha}\beta s}\int_{\beta}^{\mu} G_n(n^{\alpha}x)dx . \end{aligned}$$

But

3.1.16 $$\begin{aligned} P\left\{\left|\frac{S_n}{n^{\alpha}} - \mu\right| > \varepsilon\right\} &\leqq \frac{E(S_n - n^{\alpha}\mu)^2}{n^{2\alpha}\varepsilon^2} \\ &\leqq \frac{n[1 + (n - 1)\rho]}{n^{2\alpha}\varepsilon^2} . \end{aligned}$$

The right hand side of 3.1.16 tends to zero as $n \to \infty$. Thus $F_n(n^\alpha x) \to 0$ as $n \to \infty$ for all $x < \mu$. Hence using the mean value theorem we may write

3.1.17 $$K_n = n^\alpha e^{-n^\alpha \beta s} \delta'_n ,$$

where $\delta'_n \to 0$ as $n \to \infty$ uniformly in $s > 0$. Combining 3.1.14 and 3.1.17 and putting $\delta''_n = \delta_n - \delta'_n$, we have

3.1.18 $$\sum_{n=1}^{\infty} a_n(L_n - K_n) = \sum_{n=1}^{\infty} a_n n^\alpha e^{-n^\alpha \beta s} \delta''_n ,$$

where $\delta''_n \to 0$ as $n \to \infty$.
In view of 3.1.11 and 2.13

3.1.19 $$\frac{\sum_{n=1}^{\infty} a_n(L_n - K_n)}{s^{-1} \sum_{n=1}^{\infty} a_n e^{-n^\alpha \mu s}} \to 0, \qquad s \to 0^+ .$$

Hence

3.1.20 $$\phi(s) \sim \frac{\Gamma[(\lambda + 1)/\alpha] s^{-\{[(\lambda+1)/\alpha]+1\}} L(1/s^\alpha)}{\alpha \mu^{(\lambda+1)/\alpha}} , \qquad s \to 0^+ .$$

Using Karamata's Tauberian theorem, we have

3.1.21 $$\frac{1}{L(x^\alpha) x^{[(\lambda+1)/\alpha]+1}} \int_0^x \phi(t) dt \to \frac{\Gamma[(\lambda + 1)/\alpha]}{\alpha \Gamma\{[(\lambda + 1)/\alpha] + 2\} \mu^{(\lambda+1)/\alpha}} , \qquad x \to \infty .$$

Using the same reasoning as Heyde, we have if $x > 0, 0 < \theta < 1$

3.1.22 $$\phi(\theta x)(x - \theta x) \leqq \int_{\theta x}^x \phi(t) dt \leqq \phi(x)(x - \theta x) .$$

So

3.1.23 $$\begin{aligned} \frac{1}{x^{(\lambda+1)/\alpha} L(x^\alpha)} \phi(\theta x) &\leqq \left[\frac{1}{(1 - \theta) L(x^\alpha) x^{[(\lambda+1)/\alpha]+1}}\right] \\ &\quad \times \left[\int_0^x \phi(t) dt - \int_0^{\theta x} \phi(t) dt\right] \\ &\leqq \frac{1}{x^{(\lambda+1)/\alpha} L(x^\alpha)} \phi(x) . \end{aligned}$$

Using 3.1.21 in the above inequality we have

3.1.24 $$\begin{aligned} \limsup_{x\to\infty} \frac{\phi(\theta x)}{x^{(\lambda+1)/\alpha} L(x^\alpha)} &\leqq \frac{\Gamma[(\lambda + 1)/\alpha][1 - \theta^{[(\lambda+1)/\alpha]+1}]}{(1 - \theta)\alpha \Gamma\{[\lambda + 1)/\alpha] + 2\} \mu^{(\lambda+1)/\alpha}} \\ &\leqq \liminf_{x\to\infty} \frac{\phi(x)}{x^{(\lambda+1)/\alpha} L(x^\alpha)} . \end{aligned}$$

Taking limit as $\theta \to 1$ in the right hand side and left hand side of 3.1.24

3.1.25 $$\liminf_{x\to\infty} \frac{\phi(x)}{x^{(\lambda+1)/\alpha}L(x^\alpha)} \geqq \frac{1}{(\lambda+1)\mu^{(\lambda+1)/\alpha}},$$

and

3.1.26 $$\limsup_{x\to\infty} \frac{\phi(x)}{x^{(\lambda+1)/\alpha}L(x^\alpha)} \leqq \frac{1}{(\lambda+1)\mu^{(\lambda+1)/\alpha}}.$$

Combining the two we get

3.1.27 $$\lim_{x\to\infty} \frac{\phi(x)}{x^{(\lambda+1)/\alpha}L(x^\alpha)} = \frac{1}{(\lambda+1)\mu^{(\lambda+1)/\alpha}}.$$

So

3.1.28 $$\phi(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha}L(x^\alpha)}{(\lambda+1)}, \qquad x \to \infty$$

Now put

3.1.29 $$\psi(x) = \sum_{n=1}^{\infty} a_n G_n(x)[1 - U(x - \beta n^\alpha)]$$

so that

3.1.30 $$\sum_{n=1}^{\infty} a_n G_n(x) = \phi(x) + \psi(x).$$

From 3.1.4 and 3.1.29, we have

3.1.31 $$\psi(x) \leqq \sum_{n=1}^{\infty} a_n G_n(n^\alpha\beta) \leqq \sum_{n=1}^{\infty} a_n F_n(n^\alpha\beta) < \infty.$$

Hence

3.1.32 $$\frac{\psi(x)}{(x/\mu)^{(\lambda+1)/\alpha}L(x^\alpha)} \to 0, \qquad x \to \infty.$$

Thus

3.1.33 $$\sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha}L(x^\alpha)}{(\lambda+1)}.$$

This proves Theorem 3.1.

In the next theorem we discuss the case when x_n is a sequence of identically distributed random variables having an exponential auto-correlation.

THEOREM 3.2. *Let* $\{x_i\}$, $i = 1, 2, \cdots$ *be a sequence of identically distributed random variables with* $E(x_i) = \mu$, $i = 1, 2, \cdots$. *Let this sequence be such that the correlation between* x_i *and* x_j *is* $\rho_{ij} = \rho^{|i-j|}$, $i, j = 1, 2, \cdots$ *and* $0 < \rho < 1$. *If*

3.2.1 $$1 - H_n(nx) \leqq p(n, x) ,$$

where

3.2.2 $$\delta_n = \int_\mu^\infty p(n, x)dx \to 0, \qquad n \to \infty ,$$

the nonnegative constants $\{a_n\}$ *satisfy* 2.11 *and*

3.2.3 $$\sum_{n=1}^\infty a_n F_n(n\beta) < \infty , \qquad 0 < \beta < \mu ,$$

then

3.2.4 $$\sum_{n=1}^\infty a_n G_n(x) \sim \frac{(x/\mu)^{(\lambda+1)} L(x)}{(\lambda + 1)} .$$

Proof of Theorem 3.2. Let

3.2.5 $$\phi(x) = \sum_{n=1}^\infty a_n G_n(x) U(x - n\beta) .$$

Using the same technique as in Theorem 3.1, we have

3.2.6 $$\phi(s) = s^{-1} \sum_{n=1}^\infty a_n e^{-n\mu s} - \sum_{n=1}^\infty a_n(L_n - K_n) ,$$

where

3.2.7 $$L_n = \int_{\mu n}^\infty e^{-sx}[1 - G_n(x)]dx , \qquad K_n = \int_{\beta n}^{\mu n} e^{-sx} G_n(x)dx .$$

Using 2.14

3.2.8 $$s^{-1} \sum_{n=1}^\infty a_n e^{-n\mu s} \sim \frac{\Gamma(\lambda + 1)s^{-(\lambda+2)} L(1/s)}{\mu^{(\lambda+1)}} . \qquad s \to 0^+ .$$

Also

3.2.9 $$L_n \leqq ne^{-ns\beta} \int_\mu^\infty [1 - G_n(nx)]dx .$$

Using 3.2.1 and the fact that $G_n(x) \leqq F_n(x)$, we get

3.2.10 $$\int_\mu^\infty [1 - G_n(nx)]dx \to 0 , \qquad n \to \infty .$$

Hence we may write

3.2.11
$$L_n = ne^{-n\beta s}\delta_n ,$$

where $\delta_n \to 0, n \to \infty$ uniformly in s > 0.

Also

3.2.12
$$K_n \leqq ne^{-n\beta s} \int_\beta^\mu G_n(nx)dx .$$

Using the fact that $G_n(nx) \leqq F_n(nx)$, the law of large numbers by virtue of which $F_n(nx) \to 0$ as $n \to \infty$ for all $x < \mu$, and the mean value theorem, we way write

3.2.13
$$K_n = ne^{-n\beta s}\delta'_n ,$$

where $\delta'_n \to 0$ as $n \to \infty$.

Combining 3.2.11 and 3.2.13 and putting $\delta''_n = \delta_n - \delta'_n$,

3.2.14
$$\sum_{n=1}^\infty a_n(L_n - K_n) = \sum_{n=1}^\infty na_n e^{-n\beta s}\delta''_n ,$$

where $\delta''_n \to 0$ as $n \to \infty$.

Using 2.15 and 3.2.8,

3.2.15
$$\frac{\sum_{n=1}^\infty a_n(L_n - K_n)}{s^{-1}\sum_{n=1}^\infty a_n e^{-n\mu s}} \to 0 \quad \text{as} \quad s \to 0^+ .$$

Now put

3.2.16
$$\psi(x) = \sum_{n=1}^\infty a_n G_n(x)[1 - U(x - \beta n)] ,$$

so that

3.2.17
$$\sum_{n=1}^\infty a_n G_n(x) = \phi(x) + \psi(x) .$$

Using 3.2.3

$$\psi(x) \leqq \sum_{n=1}^\infty a_n F_n(n\beta)$$
$$< \infty .$$

So

3.2.18
$$\frac{\psi(x)}{(x/\mu)^{(\lambda+1)}L(x)} \to 0 , \qquad x \to \infty .$$

Using the same reasoning as in Theorem 3.1, we have 3.2.4.

4. Examples. We now give a few examples to illustrate the theorems. In view of their independent interest they are given in the form of theorems.

EXAMPLE 1. We now illustrate Theorem 3.1 when the sequence $\{x_i\}$ follow normal law. The result is given in Theorem 4.1.

THEOREM 4.1. *Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of normal variables with $E(x_i) = \mu_i$ and $E(x_i - \mu_i)^2 = 1$, $i = 1, 2, \cdots$. Let this sequence be such that the correlation between x_i and x_j is $\rho, 0 < \rho < 1$, $i, j = 1, 2, \cdots, i \neq j$.*

If μ_i's satisfy 3.1.1, then 3.1.5 is true.

Proof of theorem 4.1. We first prove the case when $G_n(x) = H_n(x)$. Let

4.1.1 $$\phi(x) = \sum_{n=1}^{\infty} a_n H_n(x) U(x - \beta n^\alpha) , \qquad 0 < \beta < \mu ,$$

where $U(x)$ is defined by 3.1.7.

4.1.2 $$\phi(x) = \sum_{n=1}^{\infty} a_n U(x - \mu n^\alpha) - \sum_{n=1}^{\infty} [U(x - \mu n^\alpha) - H_n(x)] U(x - \beta n^\alpha) .$$

4.1.3 $$\phi(s) = s^{-1} \sum_{n=1}^{\infty} a_n e^{-n^\alpha \mu s} - \sum_{n=1}^{\infty} a_n (L_n - K_n) .$$

Term by term integration is justified by monotone convergence. Here

4.1.4 $$L_n = \int_{\mu n^\alpha}^{\infty} e^{-sx}[1 - H_n(x)]dx , \qquad K_n = \int_{\beta n^\alpha}^{\mu n^\alpha} e^{-sx} H_n(x) dx .$$

Now

4.1.5 $$L_n = \int_{n^\alpha \mu}^{n^\alpha \mu + kn^r} e^{-sx}[1 - H_n(x)]dx + \int_{n^\alpha \mu + kn^r}^{\infty} e^{-sx}[1 - H_n(x)]dx , \qquad k > 0 , \quad 1 < r < \alpha .$$

But

4.1.6 $$\int_{n^\alpha \mu}^{n^\alpha \mu + kn^r} e^{-sx}[1 - H_n(x)]dx \leqq kn^r e^{-\beta n^\alpha s} ,$$

and

4.1.7 $$\int_{n^\alpha \mu + kn^r}^{\infty} e^{-sx}[1 - H_n(x)]dx \leqq n^\alpha e^{-n^\alpha \beta s} \int_{\frac{n^\alpha \mu + kn^r}{n^\alpha}}^{\infty} [1 - H_n(n^\alpha x)]dx .$$

Now

$$\begin{aligned}1 - H_n(n^\alpha x) &\leqq [1 - F_1(n^\alpha x)] + [1 - F_2(n^\alpha x)] \\ &\quad + \cdots + [1 - F_n(n^\alpha x)] \\ &\leqq \{1 - \Phi(n^\alpha x - \mu_1)\} + \left\{1 - \Phi\left[\frac{n^\alpha x - (\mu_1 + \mu_2)}{\sqrt{2(1+\rho)}}\right]\right\} \\ &\quad + \cdots + \left\{1 - \Phi\left[\frac{n^\alpha x - (\mu_1 + \mu_2 + \cdots + \mu_n)}{\sqrt{n[1 + (n-1)\rho]}}\right]\right\}, \end{aligned} \tag{4.1.8}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du\,. \tag{4.1.9}$$

Hence

$$1 - H_n(n^\alpha x) \leqq n\left\{1 - \Phi\left[\frac{n^\alpha x - (\mu_1 + \mu_2 + \cdots + \mu_n)}{\sqrt{n[1 + (n-1)\rho]}}\right]\right\}. \tag{4.1.10}$$

Lemma 2 in [5, p. 166] gives

$$1 - \Phi(x) \leqq \frac{1}{\sqrt{2\pi}} x^{-1} e^{-x^2/2}\,, \qquad x > 0\,. \tag{4.1.11}$$

Using 4.1.11 in 4.1.10, for sufficiently large n, we have

$$\begin{aligned}1 - H_n(n^\alpha x) &\leqq \frac{n^{3/2}\sqrt{1 + (n-1)\rho}}{\sqrt{2\pi}(n^\alpha x - n^\alpha \mu)} e^{-\frac{(n^\alpha x - n^\alpha \mu)^2}{2n[1 + (n-1)\rho]}} \\ &\leqq \frac{n^{2-\alpha}}{\sqrt{2\pi}(x - \mu)} e^{-\frac{n^{2\alpha}(x-\mu)^2}{2n^2}}\,, \qquad n^\alpha \mu + kn^r \leqq x < \infty\,. \end{aligned} \tag{4.1.12}$$

Now

$$\begin{aligned}\int_{\frac{n^\alpha \mu + kn^r}{n^\alpha}}^{\infty} [1 - H_n(n^\alpha x)] dx &\leqq \frac{n^{2-\alpha}}{\sqrt{2\pi}} \int_{\frac{n^\alpha \mu + kn^r}{n^\alpha}}^{\infty} \frac{e^{-n^{2(\alpha-1)}(x-\mu)^2/2} dx}{(x - \mu)} \\ &\leqq \frac{n^{2-r}}{k\sqrt{2\pi}} \int_{\frac{n^\alpha \mu + kn^r}{n^\alpha}}^{\infty} e^{-n^{2(\alpha-1)}(x-\mu)^2/2} dx \\ &\leqq \frac{n^{3-\alpha-r}}{k\sqrt{2\pi}} \int_{kn^{r-1}}^{\infty} e^{-u^2/2} du\,. \end{aligned} \tag{4.1.13}$$

Using 4.1.11 to the right hand side integral in 4.1.13, we finally get

$$\int_{\frac{n^\alpha \mu + kn^r}{n^\alpha}}^{\infty} [1 - H_n(n^\alpha x)] dx \leqq \frac{n^{4-\alpha-2r}}{k^2\sqrt{2\pi}} e^{-k^2 n^{2(r-1)}/2}\,. \tag{4.1.14}$$

The right hand side in 4.1.14 tends to zero as $n \to \infty$, since $r > 1$. Thus we can write

4.1.15 $$L_n \leqq kn^r e^{-\beta n^\alpha s} + n^\alpha e^{-\beta n^\alpha s}\theta_n \ ,$$

where $\theta_n \to 0$ as $n \to \infty$. Hence we can write

4.1.16 $$L_n = n^\alpha e^{-n^\alpha \beta s}\delta_n \ ,$$

where $\delta_n \to 0$ as $n \to \infty$, uniformly in $s > 0$.
Also

$$K_n \leqq n^\alpha e^{-n^\alpha \beta s}\int_\beta^\mu H_n(n^\alpha x)dx$$

4.1.17 $$\leqq n^\alpha e^{-n^\alpha \beta s}\int_\beta^\mu F_n(n^\alpha x)dx \ .$$

But using 3.1.16 and the arguments leading to 3.1.17, we get

4.1.18 $$K_n = n^\alpha e^{-n^\alpha \beta s}\delta'_n \ ,$$

where $\delta'_n \to 0$ as $n \to \infty$, uniformly in $s > 0$.
Thus

4.1.19 $$\phi(s) \sim \frac{\Gamma[(\lambda + 1)/\alpha]s^{-\{[(\lambda+1)/\alpha]+1\}}L(1/s^\alpha)}{\alpha\mu^{(\lambda+1)/\alpha}} \ , \qquad s \to 0^+ \ .$$

Take

4.1.20 $$\Psi(x) = \sum_{n=1}^\infty a_n H_n(x)[1 - U(x - \beta n^\alpha)] \ ,$$

so that

4.1.21 $$\sum_{n=1}^\infty a_n H_n(x) = \phi(x) + \Psi(x) \ .$$

Now

4.1.22 $$\begin{aligned}\Psi(x) &\leqq \sum_{n=1}^\infty a_n H_n(n^\alpha\beta) \\ &\leqq \sum_{n=1}^\infty a_n F_n(n^\alpha\beta) \ ,\end{aligned}$$

where

4.1.23 $$\begin{aligned}F_n(n^\alpha\beta) &= \frac{1}{\sqrt{2\pi n[1 + (n - 1)\rho]}}\int_{-\infty}^{\beta n^\alpha} e^{-\frac{\left(u - \sum_i^n \mu_i\right)^2}{2n[1+(n-1)\rho]}}du \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\left(\beta n^\alpha - \sum_i^n \mu_i\right)}{\sqrt{n[1+(n-1)\rho]}}} e^{-v^2/2}dv \ .\end{aligned}$$

Since the upper limit in the integral in 4.1.23 is negative for large values of n,

4.1.24
$$F_n(n^\alpha x) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\left(\sum_1^n \mu_i - n^\alpha \beta\right)}{\sqrt{n[1+(n-1)\rho]}}}^{\infty} e^{-v^2/2} dv \ .$$

Using 4.1.11 to the right hand side of 4.1.24,

4.1.25
$$F_n(n^\alpha x) \leqq \frac{\sqrt{n[1+(n-1)\rho]}}{\sqrt{2\pi}\left(\sum_1^n \mu_i - n^\alpha \beta\right)} e^{-\frac{\left(\sum_i^n \mu_i - n^\alpha \beta\right)^2}{2n[1+(n-1)\rho]}} \ .$$

Hence

4.1.26
$$\sum_{n=1}^{\infty} a_n F_n(n^\alpha x) < \infty \ .$$

So

4.1.27
$$\frac{\Psi(x)}{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)} \longrightarrow 0 \ , \qquad x \longrightarrow \infty \ .$$

Thus

4.1.28
$$\sum_{n=1}^{\infty} a_n H_n(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)}{(\lambda+1)} \ , \qquad x \longrightarrow \infty \ .$$

If we consider $\sum_{n=1}^{\infty} a_n F_n(x)$ instead of $\sum_{n=1}^{\infty} a_n H_n(x)$, the entire analysis holds. Here in 4.1.4 L_n is given by

$$L_n = \int_{n^\alpha \mu}^{\infty} e^{-sx}[1 - F_n(x)]dx \ ,$$

and

4.1.29
$$\int_{n^\alpha \mu}^{\infty} e^{-sx}[1 - F_n(x)]dx \leqq \int_{n^\alpha \mu}^{\infty} e^{-sx}[1 - H_n(x)]dx \ .$$

This reduces the problem to the case of $H_n(x)$. Thus the theorem is proved.

EXAMPLE 2. We now illustrate Theorem 3.2 when the sequence $\{x_n\}$ follow the normal law. The result is given in Theorem 4.2.

THEOREM 4.2. *Let* $\{x_i\}$, $i = 1, 2, \cdots$ *be a sequence of identically distributed normal variables with* $E(x_i) = \mu$ *and* $E(x_i - \mu)^2 = 1$, $i = 1, 2, \cdots$. *If this sequence be such that the correlation between*

x_i and x_j is given by $\rho_{ij} = \rho^{|i-j|}$, $i, j = 1, 2, \cdots$ and $0 < \rho < 1$, then 3.2.4 is true.

Proof of Theorem 4.2. Using the same notation as in Theorem 4.1, we have

4.2.1 $$\phi(x) = \sum_{n=1}^{\infty} a_n U(x - n\mu) - \sum_{n=1}^{\infty} a_n[U(x - n\mu) - H_n(x)]U(x - n\beta) .$$

Thus

$$\phi(s) = s^{-1} \sum_{n=1}^{\infty} a_n e^{-n\mu s} - \sum_{n=1}^{\infty} a_n(L_n - K_n) ,$$

where

4.2.2 $$L_n = \int_{\mu n}^{\infty} e^{-sx}[1 - H_n(x)]dx , \qquad K_n = \int_{\beta n}^{\mu n} e^{-sx} H_n(x)dx .$$

4.2.3 $$L_n = \int_{\mu n}^{\mu n + kn^r} e^{-sx}[1 - H_n(x)]dx + \int_{n\mu + kn^r}^{\infty} e^{-sx}[1 - H_n(x)]dx , \quad k > 0, 1/2 < r < 1 .$$

Now

4.2.4 $$\int_{n\mu}^{n\mu + kn^r} e^{-sx}[1 - H_n(x)]dx \leqq kn^r e^{-n\beta s} .$$

and

4.2.5 $$\int_{n\mu + kn^r}^{\infty} e^{-sx}[1 - H_n(x)]dx \leqq ne^{-n\beta s} \int_{\frac{n\mu + kn^r}{n}}^{\infty} [1 - H_n(nx)]dx .$$

But

4.2.6 $$1 - H_n(nx) \leqq n\left\{1 - \Phi\left[\frac{n(x - \mu)}{\sqrt{\dfrac{n(1 + \rho)}{(1 - \rho)} - \dfrac{2\rho(1 - \rho^n)}{(1 - \rho)^2}}}\right]\right\} .$$

Using 4.1.11 to the right side of 4.2.6

4.2.7 $$1 - H_n(nx) \leqq \frac{n\sqrt{\dfrac{n(1 + \rho)}{(1 - \rho)} - \dfrac{2\rho(1 - \rho^n)}{(1 - \rho)^2}}}{\sqrt{2\pi}\, n(x - \mu)} e^{-\frac{n^2(x-\mu)^2}{2\left[\frac{n(1+\rho)}{(1-\rho)} - \frac{2\rho(1-\rho^n)}{(1-\rho)^2}\right]}} , \quad x > \mu .$$

Hence

$$\int_{\frac{n\mu + kn^r}{n}}^{\infty} [1 - H_n(nx)]dx \leqq \left[\frac{n\sqrt{\dfrac{n(1 + \rho)}{(1 - \rho)} - \dfrac{2\rho(1 - \rho^n)}{(1 - \rho)^2}}}{k\sqrt{2\pi}n^r}\right] \times \int_{\frac{n}{n\mu + n^r k}}^{\infty} e^{-\frac{n^2(x-\mu)^2}{2\left[\frac{n(1+\rho)}{(1-\rho)} - \frac{2\rho(1-\rho^n)}{(1-\rho)^2}\right]}} dx .$$

4.2.8
$$\leqq \frac{\left[\frac{n(1+\rho)}{(1-\rho)} - \frac{2\rho(1-\rho^n)}{(1-\rho)^2}\right]}{k\sqrt{2\pi n^r}} \int_{\frac{kn^r}{\left[\frac{n(1+\rho)}{(1-\rho)} - \frac{2\rho(1-\rho^n)}{(1-\rho)^2}\right]}}^{\infty} e^{-u^2/2} du \,.$$

Using 4.1.11 to the right hand side of 4.2.8

4.2.9
$$\int_{\frac{n\mu+kn^r}{n}}^{\infty} 1 - H_n(nx)dx \leqq \frac{\left[\frac{n(1+\rho)}{(1-\rho)} - \frac{2\rho(1-\rho^n)}{(1-\rho)^2}\right]^{3/2}}{k^2\sqrt{2\pi n^r}} e^{-\frac{n^{2r}}{2\left[\frac{n(1+\rho)}{(1-\rho)} - \frac{2\rho(1-\rho^n)}{(1-\rho)^2}\right]}} \,.$$

The expression on the right hand side of $4.2.9 \to 0$ as $n \to \infty$, since $1/2 < r < 1$. The rest of the arguments are as in the previous example and the theorem is proved.

EXAMPLE 3. We now give another example to illustrate Theorem 3.2, when the sequence $\{x_n\}$ follow the type III distribution. The result is given in Theorem 4.3.

THEOREM 4.3. *Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of identically distributed Gamma variables correlated according to an exponential auto-correlation law and that the correlation between x_i and x_j is given by $\rho_{ij} = \rho^{|i-j|}$, $i, j = 1, 2, \cdots$ and $0 < \rho < 1$. Let*

$$P(x_i \leqq x) = \theta^{-r}[\Gamma(r)]^{-1} e^{-x/\theta} x^{r-1} \,, \qquad x \geqq 0 \,,$$
$$= 0 \,, \qquad x < 0 \,,$$
$$i = 1, 2, \cdots \,.$$

Then

4.3.1
$$\sum_{n=1}^{\infty} a_n F_n(x) \sim \frac{(x/r\theta)^{\lambda+1} L(x)}{(\lambda+1)} \,, \qquad x \to \infty$$

and

4.3.2
$$\sum_{n=1}^{\infty} a_n P(x < S_n \leqq x + h) \sim \frac{hL(x)}{r\theta} (x/r\theta)^{\lambda} \,, \qquad h > 0, x \to \infty \,,$$

where the a_n's satisfy 2.11.

Proof of Theorem 4.3. Using the results of Samuel Kotz and John W. Adams, $\phi_n(t)$, the characteristic function of the distribution of the sum S_n is

4.3.3
$$\phi_n(t) = \prod_{j=1}^{n} (1 - it\theta\mu_j)^{-r} \,,$$

where

4.3.4 $\mu_j = (1 - 2\sqrt{\rho}\cos\theta_j + \rho)^{-1}(1-\rho), \qquad j = 1, 2, \cdots .$

Here θ_j's are the values of θ which satisfy one or other of the equations

4.3.5
$$\begin{aligned}\sin[(n+1)\theta/2] &= \sqrt{\rho}\sin[(n-1)\theta/2] ,\\ \cos[(n+1)\theta/2] &= \sqrt{\rho}\cos[(n-1)\theta/2] .\end{aligned}$$

Let

$$H(x) = \sum_{n=1}^{\infty} a_n F_n(x)$$

and

$$H(s) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-sx} dF_n(x) .$$

Using 4.3.3

4.3.6
$$\begin{aligned}H(s) &= \sum_{n=1}^{\infty} a_n \prod_{j=1}^{n} (1 + s\theta\mu_j)^{-r}\\ &= \sum_{n=1}^{\infty} a_n e^{-r\sum_{j=1}^{n}\log(1+s\theta\mu_j)} .\end{aligned}$$

Using the fact that $\log(1+z) = z + \lambda z^2$, $|\lambda| < 1$, $|z| < 1/2$, we write

4.3.7 $\log(1 + s\theta\mu_j) = s\theta\mu_j + \lambda_j s^2\theta^2\mu_j^2 , \qquad |\lambda_j| < 1, j = 1, 2, \cdots .$

Also $[(1+\sqrt{\rho})/(1-\sqrt{\rho})]$ is the maximum value of μ_j and $\sum_{j=1}^{n}\mu_j = n$. Using these we get

4.3.8 $\sum_{j=1}^{\infty}\log(1 + s\theta\mu_j) = s\theta n + [s^2\theta^2\mu n(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2] , \ |\mu| < 1 .$

Using this in 4.3.6, we get

4.3.9
$$\begin{aligned}H(s) &= \sum_{n=1}^{\infty} a_n e^{-r\theta n s} e^{-r\mu n s^2\theta^2(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2} .\\ &= \sum_{n=1}^{\infty} a_n e^{-r\theta n s}[e^{-\mu r n s^2\theta^2(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2} - 1 + 1] .\\ &= I_1 + I_2 (\text{say}) .\end{aligned}$$

4.3.10
$$I_1 = \sum_{n=1}^{\infty} a_n e^{-r\theta n s} .$$

Using 2.14

4.3.11 $I_1 \sim \Gamma(\lambda+1)(sr\theta)^{-(\lambda+1)}L(1/s) , \qquad s \to 0^+ .$

Now

4.3.12 $$I_2 = \sum_{n=1}^{\infty} a_n e^{-r\theta ns}[e^{-r\mu n\theta^2 s^2(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2} - 1] .$$

Since $e^x - 1 < |x| e^{|x|}$, we get

$$|I_2| < \sum_{n=1}^{\infty} a_n e^{-r\theta ns} r |\mu| ns^2\theta^2[(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2] e^{r|\mu|n\theta^2 s^2 \frac{(1+\sqrt{\rho})^2}{(1-\sqrt{\rho})^2}} .$$

4.3.13 $$\leqq r|\mu| s^2\theta^2[(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2] \sum_{n=1}^{\infty} a_n n e^{-r\theta ns[1-p(s)]} ,$$

where $p(s)$ can be made as small as we like since $s \to 0^+$. Thus using 2.15,

4.3.14 $$|I_2| \leqq r|\mu| s^2\theta^2[(1+\sqrt{\rho})^2/(1-\sqrt{\rho})^2]\Gamma(\lambda+2)(r\theta s)^{-(\lambda+2)}L(1/s) , \quad s \to 0^+ .$$

Hence

4.3.15 $$|I_2|/I_1 \to 0 \quad \text{as} \quad s \to 0^+ .$$

Using this we get

$$H(s) \sim \Gamma(\lambda+1)(sr\theta)^{-(\lambda+1)}L(1/s) .$$

By Karamata's Tauberian theorem, we get 4.3.1. This proves the first part of the theorem.

To prove the second part of the theorem, take

4.3.16 $$\begin{aligned} Q(x) &= \sum_{n=1}^{\infty} a_n P(x < S_n \leqq x+h) \\ &= \sum_{n=1}^{\infty} a_n[F_n(x+h) - F_n(x)] . \end{aligned}$$

Let

4.3.17 $$Q(s) = \int_0^{\infty} e^{-sx}\, dQ(x) .$$

Then

4.3.18 $$\begin{aligned} Q(s) &= \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-sx} d[F_n(x+h) - F_n(x)] . \\ &= \sum_{n=1}^{\infty} a_n(e^{sh}-1)\int_0^{\infty} e^{-sx}dF_n(x) - \sum_{n=1}^{\infty} a_n \int_0^{h} e^{-sx}dF_n(x) . \end{aligned}$$

Now

4.3.19 $$\sum_{n=1}^{\infty} a_n(e^{sh}-1)\int_0^{\infty} e^{-sx}dF_n(x) \sim (h/r\theta)\Gamma(\lambda+1)(sr\theta)^{-\lambda}L(1/s) , \quad s \to 0^+ .$$

Also

4.3.20 $$\int_0^h e^{-sx} dF_n(x) \leqq F_n(h) .$$

So

4.3.21 $$\sum_{n=1}^{\infty} a_n \int_0^h e^{-sx} dF_n(x) \leqq \sum_{n=1}^{\infty} a_n F_n(h) .$$

But we can show that

4.3.22 $$\sum_{n=1}^{\infty} n^k P\{|S_n - nr\theta| > n\varepsilon\} < \infty , \qquad k > 0 .$$

Hence

4.3.23 $$\sum_{n=1}^{\infty} n^k F_n(x) \leqq \sum_{n=1}^{\infty} n^k P\{S_n \leqq n(r\theta - \varepsilon)\} < \infty .$$

Using this in 4.3.21, we have

4.3.24 $$\sum_{n=1}^{\infty} a_n \int_0^h e^{-sx} dF_n(x) < \infty .$$

Hence

4.3.25 $$\frac{\sum_{n=1}^{\infty} a_n \int_0^h e^{-sx} dF_n(x)}{sh\Gamma(\lambda + 1)(sr\theta)^{-(\lambda+1)} L(1/s)} \longrightarrow 0 \quad \text{as} \quad s \longrightarrow 0^+ .$$

Using 4.3.19 and 4.3.25, we have

4.3.26 $$Q(s) \sim (h/r\theta)\Gamma(\lambda + 1)(sr\theta)^{-\lambda} L(1/s) .$$

Using Karamata's Tauberian theorem we get 4.3.2. This proves the second part of the theorem.

In particular if $a_n = 1$, then

4.3.27 $$Q(x) = \sum_{n=1}^{\infty} P(x < S_n \leqq x + h) \sim h/r\theta = h/E(x_i) .$$

This is in agreement with the classical renewal theorem. We remark that in the case of exponentially auto-correlated Gamma variables, the asymptotic behaviour of $Q(x)$ is independent of the correlotion coefficient and hence is same as if $\rho = 0$ and the variables are independent.

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