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MEAN VALUE ITERATION OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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This paper applies a certain method of iteration, of the mean value type introduced by W. R. Mann, to obtain two theorems on the approximation of a fixed point of a mapping of a Banach space into itself which is nonexpansive (i.e., a mapping which satisfies $\|Tx - Ty\| \leqq \|x - y\|$ for each x and y).

The first theorem obtains convergence of the iterates to a fixed point of a nonexpansive mapping which maps a compact convex subset of a rotund Banach space into itself.

The second theorem obtains convergence to a fixed point provided that the Banach space is uniformly convex and the iterating transformation is nonexpansive, maps a closed bounded convex subset of the space into itself, and satisfies a certain restriction on the distance between any point and its image.

We note that a rotation T about zero of the closed unit disc in the complex plane satisfies the conditions of Theorems 1 and 2, but the usual sequence $\{T^n x\}$ of iterates of x does not converge unless x is zero.

DEFINITIONS. If Y is a Banach space, T is a mapping from Y into itself, and $x \in Y$, then $M(x, T)$ is the sequence $\{v_n\}$ defined by $v_1 = x$ and $v_{n+1} = (1/2)(v_n + Tv_n)$.

Following Wilansky [3, pp. 107–111], we say that a Banach space Y is *rotund* provided that if $w \in Y$, $y \in Y$, $w \neq y$, and $\|w\| = \|y\| \leqq 1$, then $(1/2)\|w + y\| < 1$.

THEOREM 1. *Let Y be a rotund Banach space, E be a compact convex subset of Y, and T be a nonexpansive mapping which maps E into itself. If $x \in E$ then $M(x, T)$ converges to a fixed point of T.*

Proof. If, for some n, $v_n = Tv_n$, then clearly $M(x, T)$ converges to v_n.

Hence suppose that $v_n \neq Tv_n$, for each n. Let z be a fixed point of T. Then $\{\|v_n - z\|\}$ is decreasing, for since Y is rotund and

$$\|Tv_n - z\| = \|Tv_n - Tz\| \leqq \|v_n - z\| ,$$

we have that

$$\|v_{n+1} - z\| = \left\| \frac{1}{2}(v_n + Tv_n) - z \right\| < \|v_n - z\| .$$

Suppose that $\lim_n \| v_n - z \| = b > 0$. Let y be a cluster value of $\{v_n\}$. Then clearly $b = \| y - z \|$.

Suppose first that $y = Ty$. Then for each n,

$$\| Tv_n - y \| = \| Tv_n - Ty \| \leqq \| v_n - y \| .$$

Since we have assumed that $v_n \neq Tv_n$ for each n, we have by the rotundity of Y that

$$\| v_{n+1} - y \| = \left\| \frac{1}{2}(v_n + Tv_n) - y \right\| < \| v_n - y \| .$$

Thus $\{\| v_n - y \|\}$ is decreasing, and since y is a cluster value of $\{v_n\}$, $M(x, T)$ converges to y.

Now suppose that $y \neq Ty$. Let d denote $b - \| (1/2)(y + Ty) - z \|$. Then $d > 0$, since Y is rotund, for

$$\| Ty - z \| = \| Ty - Tz \| \leqq \| y - z \| = b .$$

Let n be such that $\| y - v_n \| < d$. Then since T is nonexpansive,

$$\begin{aligned} \left\| \frac{1}{2}(y + Ty) - v_{n+1} \right\| &= \left\| \frac{1}{2}(y + Ty) - \frac{1}{2}(v_n + Tv_n) \right\| \\ &\leqq \frac{1}{2} \| y - v_n \| + \frac{1}{2} \| Ty - Tv_n \| \\ &\leqq \| y - v_n \| < d . \end{aligned}$$

Hence

$$\begin{aligned} \| v_{n+1} - z \| &\leqq \left\| v_{n+1} - \frac{1}{2}(y + Ty) \right\| + \left\| \frac{1}{2}(y + Ty) - z \right\| \\ &< d + (b - d) = b , \end{aligned}$$

a contradiction. Therefore $b = \lim_n \| v_n - z \| = 0$, so that $M(x, T)$ converges to z.

F. E. Browder [1] has shown that each nonexpansive mapping which maps a closed bounded convex subset E of a uniformly convex Bananch space into itself has a fixed point in E.

If such a mapping satisfies one additional requirement, we may approximate one of its fixed points using $M(x, T)$:

THEOREM 2. *Let Y be a uniformly convex Banach space, E be a closed bounded convex subset of Y, and let T be a nonexpansive mapping which maps E into itself. Let F denote the set of fixed point of T in E, and suppose that there is a number c in* $(0, 1)$ *such that if $x \in E$, then*

$$\| x - Tx \| \geqq cd(x, F) ,$$

where $d(x, F)$ *denotes* $\sup_{z \in F} \| x - z \|$.

If $x \in E$ *then* $M(x, T)$ *converges to a fixed point of* T.

Proof. The theorem is trivial if $x \in F$. Suppose that $x \in E - F$ and that $M(x, T)$ does not converge to a member of F. Then $v_n \notin F$ for each n. Since Y is uniformly convex, we have as in the proof of Theorem 1 that if $z \in F$ then $\{\| v_n - z \|\}$ is decreasing.

Suppose that $b = \lim_n d(v_n, F) > 0$. Since Y is uniformly convex, there is an r in $(0, 2b)$ such that, for w, y, and z in Y, the relations

$$\| w - z \| \leqq \| y - z \| \leqq 2b \quad \text{and} \quad \| w - y \| \geqq cb$$

imply that

$$\left\| \frac{1}{2}(w + y) - z \right\| \leqq \| y - z \| - r .$$

There is a positive integer n and a member z of F such that

$$\| v_n - z \| < b + \frac{r}{2} ,$$

so that since

$$\| Tv_n - z \| = \| Tv_n - Tz \| \leqq \| v_n - z \| < 2b$$

and

$$\| Tv_n - v_n \| \geqq cd(v_n, F) \geqq cb ,$$

we have that

$$\begin{aligned} \| v_{n+1} - z \| &= \left\| \frac{1}{2}(v_n + Tv_n) - z \right\| \\ &\leqq \| v_n - z \| - r < b + \frac{r}{2} - r < b , \end{aligned}$$

an contradiction. Hence $\lim_n d(v_n, F) = 0$.

We now need the following:

LEMMA. *If* $s > 0, z \in F$, *and* $r > 0$ *such that for some* n, v_n *is in the open sphere* $S(z, r)$ *with center* z *and radius* r, *then there exist* t *in* $(0, s)$, w *in* F, *and an* m *such that the closed sphere* $\bar{S}(w, t)$ *lies in* $S(z, r)$, *and for each* $p, v_{m+p} \in S(w, t)$.

Proof. Recall that $\{\| v_p - z \|\}$ is decreasing and that we are supposing that $\{v_p\}$ does not converge to z. Let $a = \lim_p \| v_p - z \|$.

Then $0 < a < r$. Let $t = (1/3) \min \{r - a, s\}$.

Since $\lim_p \| v_p - z \| = a$, $\lim_p d(v_p, F) = 0$, and $v_p \notin F$ for each p, there exist w in F and an m such that $\| v_m - z \| < a + t$ and $\| v_m - w \| < t$.

Since $w \in F$, $\| v_{m+p} - w \|$ decreases as p increases, so that $v_{m+p} \in S(w, t)$ for each p. Also, if $y \in \bar{S}(w, t)$, then $y \in S(z, r)$, for

$$\begin{aligned} \| y - z \| &\leqq \| y - w \| + \| w - v_m \| + \| v_m - z \| \\ &< t + t + (a + t) \\ &\leqq 3\left(\frac{r - a}{3}\right) + a = r \, . \end{aligned}$$

The lemma guarantees the existence of a sequence $\{z_i\}$ in F, a sequence $\{t_i\}$ of positive numbers with limit 0, and a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that for each i and each p,

$$\bar{S}(z_{i+1}, t_{i+1}) \quad \text{lies in} \quad S(z_i, t_i)$$

and

$$v_{n_i+p} \in S(z_i, t_i) \, .$$

By the Cantor Intersection Theorem, $\bigcap_{i=1}^{\infty} S(z_i, t_i)$ contains exactly one point, say w. Clearly $\{z_i\}$ converges to w and $w \in F$. Further, $\{\| v_n - w \|\}$ is decreasing and $\{v_{n_i}\}$ converges to w, so that $\{v_n\}$ converges to w. Thus we have contradicted our assumption that $M(x, T)$ does not converge to a member of F.

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proceedings of the National Academy of Sciences, U. S. A., **54** (1965) 1041-1044.

2. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

3. A. Wilansky, *Functional analysis*, Blaisdell, New York, (1964).

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