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CELL-LIKE MAPPINGS, I

R. C. LACHER

Cell-like mappings are introduced and studied. A space is cell-like if it is homeomorphic to a cellular subset of some manifold. A mapping is cell-like if its point-inverses are cell-like spaces. It is shown that proper, cell-like mappings of ENR'S (Euclidean NR's) form a category which includes both proper, contractible maps of ENR's and proper, cellular maps from manifolds to ENR's. It is difficult to break out of the category: The image of a proper, cell-like map on an ENR, is again an ENR, provided the image is finite-dimensional and Hausdorff.

Some applications to (unbounded) manifolds are given. For example: A cell-like map between topological manifolds of dimension $\geqq 5$ is cellular. The property of being an open n-cell, $n \geqq 5$, is preserved under proper, cell-like maps between topological manifolds. The image of a proper, cellular map on an n-manifold is a homotopy n-manifold.

The concept of cell-like mappings of ENR's extends the idea of cellular maps of manifolds by allowing point-inverses to be embeddable, rather than embedded, as cellular sets. This change in viewpoint has several advantages. First, one can study cell-like maps on manifolds and ask when such maps are cellular; this direction of study will presumably clarify certain aspects of the theory of cellular maps on (or decompositions of) manifolds. Second, and perhaps more important, in replacing the old setting with the more general one, much better results on the homotopy structure of maps become apparent. A third advantage is that the new concept generalizes several old concepts at once, and hence is, in a sense, unifying.

The most interesting types of mappings which fall into the "cell-like maps of ENR's" category are the cellular maps on manifolds and the contractible maps of ENR's. Cellular maps on manifolds (or cellular decompositions of manifolds) have been studied extensively, yet our results for cell-like mappings yield new results in this fields. Piecewise linear contractible mappings of piecewise linear manifolds have also been studied, notably in [7] and [8]. Contractible maps of ANR's were studied by Smale in [27]. Smale's conclusions (for contractible maps of ENR's) are improved here.

The primary purpose of this first paper is to develop the basic homotopy properties of cell-like mappings. Some applications to manifolds and related results are given in the last section. Other applications to manifolds will be given in a latter paper.

We rarely consider maps defined on spaces more general than ENR's (three exceptions: (2.3), (3.1), and (3.4)), although many arguments would go through under less restriction. In this sense, the emphasis is placed on strengthening conclusions rather than weakening hypotheses.

Preliminaries. To avoid confusion, we present here some definitions and conventions. R^n is euclidean n-space. B^n is the closed unit ball in R^n. S^n is the boundary of B^{n+1}. $I = [0, 1]$. An *n-cell* (resp. *open n-cell, n-sphere*) is a space homeomorphic to B^n (resp. R^n, S^n).

An *n-manifold* is a separable metric space N which is locally euclidean; i.e., each point of N has an open n-cell neighborhood. (By *neighborhood of A in X* we always mean an open subset of X which contains A.) An *n-manifold with boundary* is a separable metric space in which each point has a neighborhood whose closure is an n-cell. If N is an n-manifold with boundary, Int N denotes the subset of points having open n-cell neighborhoods, and $\operatorname{Bd} N = N - \operatorname{Int} N$.

An ENR is a space homeomorphic to a retract of an open subset of some euclidean space. Basic references for ANR's are [3] and [13]. The following explains the relationship between ANR and ENR.

LEMMA. *A metric space is an* ENR *if and only if it is a locally compact, finite-dimensional* ANR.

The proof is not difficult using [14].

SYMBOLS. We use $\pi_0 X$ to denote the path components of X and $\pi_q X$, $q \geqq 1$, to denote the q-th homotopy group of X. (Whenever we use these symbols, it will be clear what to do about base points.) We use the symbol "$\approx$" to mean "is homeomorphic to."

1. Cell-like mappings of ENR's. A *space A is cell-like* if there are a manifold M and an embedding $\varphi: A \to M$ such that $\varphi(A)$ is cellular in M; i.e., $\varphi(A)$ is the intersection of a sequence $Q_1, Q_2, \cdots$ of (closed) m-cells in M, where $Q_{i+1} \subset \operatorname{Int} Q_i$ for each i and $m = \dim M$. (See [5].) A *mapping $f: X \to Y$ is cell-like* if $f^{-1}(y)$ is a cell-like space for each $y \in Y$.

Cell-like spaces and mappings were introduced in [19] and [20] as the natural generalization of cellular sets and cellular mappings of manifolds. (In fact, the results announced in [20] are proved in this paper.) We would now like to state the main result of [19]; in order to do this we need the following property (called Property (**) in [19] and [20]).

Property UV^∞. An embedding $\varphi: A \to X$ has Property UV^∞ if, for each open set U of X containg $\varphi(A)$, there is an open set V of X, with $\varphi(A) \subset V \subset U$, such that the inclusion $V \subset U$ is null-homotopic in U.

THEOREM 1.1. *Let A be a nonvoid finite-dimensional compact metric space. Then the following conditions are equivalent*:

(a) *A is cell-like.*

(b) *A has the Čech-homotopy-type (or "fundamental shape") of a point* (see [4]).

(c) *There exists an embedding of A into some* ENR *which has Property UV^∞.*

(d) *Any embedding of A into any* ANR *has Property UV^∞.*

Theorem 1.1 is a direct quote of [19]. In our applications below, we will not need the equivalence of (a) and (b). However, the equivalence of (a), (c), and (d) is used repeatedly, often without reference to Theorem 1.1.

Recall that a *proper* mapping is one under which preimages of compact sets are compact. Two proper maps $h_0, h_1: X \to Y$ are *properly homotopic* if there is a proper map $h: X \times I \to Y$. Finally, a *proper homotopy equivalence*: $X \to Y$ is a proper map $f: X \to Y$ for which there exists a proper map $g: Y \to X$ such that the compositions $gf: X \to X$ and $fg: Y \to Y$ are properly homotopic to the appropriate identity maps.

THEOREM 1.2. *Let X and Y be* ENR's, *and let f be a proper mapping of X onto Y. Then the following are equivalent*:

(a) *f is cell-like.*

(b) *If $V \subset U$ are open sets in Y, with V contractible in U, then $f^{-1}(V)$ is contractible in $f^{-1}(U)$.*

(c) *For any open subset U of Y, $f \mid f^{-1}(U): f^{-1}(U) \to U$ is a proper homotopy equivalence.*

REMARKS. (1) The "proper" conclusion in (c) is very important. We will use this property (in a later paper) to show that the number (and homotopy type) of ends of manifolds is preserved under cell-like maps. This in turn will allow the proof of invariance of several important non-compact topological types under cell-like mappings.

(2) D. Sullivan has proved that a map $f: M \to N$ of closed PL manifolds which satisfies (c) is homotopic to a PL isomorphism: $M \to N$, provided $\dim M \geqq 5$ and $\pi_1(M) = H^3(M; Z_2) = 0$. For a proof of this remarkable generalization of the hauptvermutung, see [25]. We will not use this fact here.

(3) In § 2, condition (b) will be "weakened" in two senses. See (2.2).

For the proof of (1.2), notice that (c) obviously implies (b). Moreover, using (1.1), it is easy to see that (b) implies (a). The fact that

(a) implies (c) will be proved in § 2. (See Theorem 2.1.)

COROLLARY 1.3. *A proper, cell-like map of* ENR'*s is a proper homotopy equivalence.*

(Notice that a cell-like mapping is necessarily onto, since the empty space is *not* cell-like.)

The next result says essentially that ENR's and cell-like maps form a category.

THEOREM 1.4. *Let $f: X \to Y$ be a proper, cell-like map of* ENR'*s, and let A be a subset of Y. Then A is cell-like if and only if $f^{-1}(A)$ is cell-like.*

Proof. The inclusion $f^{-1}(A) \subset X$ has Property UV^∞ if and only if the inclusion $A \subset Y$ has the same property, by (1.2). The result follows from (1.1).

THEOREM 1.5. *The* (*Tychonoff*) *product of two cell-like spaces* (*or maps*) *is again a cell-like space* (*map*).

Proof. If A and B are cell-like, then there are manifolds M, N and embeddings $\varphi: A \to M$, $\psi: B \to N$ such that $\varphi(A)$ is cellular in M and $\psi(B)$ is cellular in N. Obviously, $\varphi(A) \times \psi(B) = (\varphi \times \psi)(A \times B)$ is cellular in $M \times N$.

We conclude this section with the observation that an onto, proper map with contractible point-inverses is cell-like.

THEOREM 1.6. *A contractible, finite-dimensional, compact metric space is cell-like.*

Proof. Let A be such a space. Then we may as well assume that $A \subset R^n$ for some n. (See [14].) Let $r: A \times I \to A$ be a map such that $r_0 =$ identity and $r_1(A) =$ point. Let U be a neighborhood of A in R^n, and define B and $\bar{r}: B \to U$ as follows:

$$B = (A \times I) \cup (U \times \{0, 1\}) ,$$
$$\bar{r} | A \times I = r, \bar{r}_0 = \text{identity}, \bar{r}_1(U) = \text{point} .$$

Since U is an ANR, there is a neighborhood of B in $U \times I$ over which $\bar{r}$ can be extended. Hence, there in a neighborhood V of A in U and a map R: $V \times I \to U$ such that $R_0 =$ inclusion and $R_1(V) =$ point. I.e., the inclusion $V \subset U$ is null-homotopic in U. Therefore the inclusion $A \subset R^n$ has Property UV^∞ and A is cell-like.

REMARK. Most cell-like spaces are not contractible or locally connected. See Theorem 2 of [2], and note that a pseudo-arc is cell-like.

2. Mapping theorems for proper homotopy.

In this section we will need the following weak versions of Property UV^∞ and "cell-like mapping."

Property UV^k. An embedding $\varphi: A \to X$ has Property UV^k if, for each open set U of X containing $\varphi(A)$, there is an open set V of X, with $\varphi(A) \subset V \subset U$, such that any map $S^q \to V$ can be extended to a map $B^{q+1} \to U$, $0 \leqq q \leqq k$.

UV^k-Trivial maps. A map $f: X \to Y$ is *UV^k-trivial* if the inclusion $f^{-1}(y) \subset X$ has Property UV^k for each $y \in Y$.

We will eventually prove that a UV^k trivial map of ENR's is cell-like provided that k exceeds the simplicial dimension of the ENR's according to the following definition.

Simplicial dimension. The *simplicial dimension* $\mathrm{sd}\, X$ of a space X is the smallest integer k such that X embeds into a locally finite k-complex. If X and Y are spaces, we define $\mathrm{sd}\,(X, Y)$ to be $\mathrm{sd}\,(X \times 0 \cup Y \times 1) = \max\{\mathrm{sd}\, X, \mathrm{sd}\, Y\}$.

Some relations between simplicial dimension and ordinary dimension.

(1) If X is a metric space then $\dim X \leqq \mathrm{sd}\, X \leqq 2 \dim X + 1$. (See [14].)

(2) If P is a locally finite polyhedron, then $\mathrm{sd}\, P = \dim P$.

(3) *Product formula.* If X, Y are metric spaces,

$$\mathrm{sd}\,(X \times Y) \leqq \mathrm{sd}\, X + \mathrm{sd}\, Y\,.$$

The main result of this section is the following:

THEOREM 2.1. *Let X and Y be* ENR's, *and let f be a proper, UV^{k-1}-trivial mapping of X onto Y, where $k = \mathrm{sd}\,(X \times I, Y)$. Then, for any open subset U of Y, $f|f^{-1}(U): f^{-1}(U) \to U$ is a proper homotopy equivalence.*

Clearly Theorem 2.1 implies Theorem 1.2, since a cell-like map of ENR's is UV^∞-trivial. (See Theorem 1.1.) Also, since an onto, UV^k-trivial map of ENR's is again an onto, UV^k-trivial map of ENR's when restricted to an inverse open set, we can apply Theorem 2.1 to get the promised "weakening" of 1.2(b), as follows.

COROLLARY 2.2. *Let X and Y be* ENR's, *and let f be a mapping of X onto Y. Then the following conditions are equivalent*:

(a) *f is cell-like.*

(b) *f is UV^k-trivial for some $k \geqq \mathrm{sd}(X \times I, Y) - 1$.*

(c) *If $V \subset U$ are sufficiently small open sets in Y, with V contractible in U, then $f^{-1}(V)$ is contractible in $f^{-1}(U)$.*

REMARK. Condition (c) of (2.2) obviously implies that f is UV^∞-trivial.

Before beginning the proof of (2.1), we will prove the following fact.

LEMMA 2.3. *Let the following be given*:

(i) *Locally compact metric spaces X and Y.*

(ii) *A locally finite k-complex K with a locally finite subcomplex L.*

(iii) *A proper UV^{k-1}-trivial mapping f of X onto Y.*

(iv) *A proper map $\varphi: K \to Y$.*

(v) *A proper map $\psi: L \to X$ such that $f\psi = \varphi | L$.*

(vi) *A continuous function $\varepsilon: Y \to (0, \infty)$.*

(vii) *A metric d on X and Y under which closed, bounded sets are compact.*

Then, there exists a proper map $\bar{\varphi}: K \to X$ such that $\bar{\varphi} | L = \psi$ and $d(f\bar{\varphi}, \varphi) \leqq \varepsilon\varphi$.

Proof. For each $y \in Y$, let U_y be a neighborhood of y with

$$\operatorname{diam} U_y \leqq \min \varepsilon | U_y$$

and

$$\operatorname{diam} f^{-1}(U) \leqq \operatorname{diam} f^{-1}(y) + 1 .$$

Then, in particular, U_y and $f^{-1}(U_y)$ have compact closures. Let V_y be another neighborhood of y, with $\bar{V}_y \subset U_y$, such that any singular q-sphere in $f^{-1}(V_y)$ is contractible in $f^{-1}(U_y)$, $0 \leqq q < k$. Finally, let $\gamma: Y \to (0, \infty)$ be a continuous function such that, if the diameter of a compact set S is no greater than $\max \gamma | S$, then $S \subset V_y$ for some y.

Now, we can easily prove the lemma assuming $\dim(K - L) = 0$. Therefore, we may assume inductively that $K - L$ is p-dimensional and that the lemma is true for the $(p-1)$-skeleton of any subdivision of K and any function ε. Choose a subdivision K_1 of K so that

$$\operatorname{diam} \varphi(\sigma) \leqq \frac{1}{2} \min \gamma\varphi \,|\, \sigma$$

for each $\sigma \in K_1$, and assume that $\psi: K_1^{p-1} \cup L_1 \to X$ is a proper map such that

$$d(f\psi, \varphi \,|\, K_1^{p-1} \cup L_1) \leqq \frac{1}{2}\gamma\varphi .$$

We extend ψ over the p-simplices of $K_1 - L_1$ one at a time, (assuming that $p \leqq k$).

Let σ be a p-simplex of $K_1 - L_1$. Then $f\psi(\operatorname{Bd} \sigma) \cup \varphi(\sigma)$ has diameter no greater than $\max \gamma\varphi(\sigma)$, and hence lies in V_y for some $y \in Y$; in particular, $\psi(\operatorname{Bd} \sigma) \subset f^{-1}(V_y)$. By Property UV^{p-1}, we can extend $\psi \,|\, \operatorname{Bd} \sigma$ to a map $\psi_\sigma: \sigma \to f^{-}(U_y)$. It follows immediately that $d(f\psi_\sigma, \varphi \,|\, \sigma) \leqq \varepsilon\varphi \,|\, \sigma$, since both $f\psi_\sigma(\sigma)$ and $\varphi(\sigma)$ lie in U_y. The union of the ψ_σ extends ψ over $K_1^p - L_1$. It is easy to check that ψ is a proper map, using the condition that $\operatorname{diam} f^{-1}(U_y) \leqq \operatorname{diam} f^{-1}(y) + 1$.

Proof of (2.1). We may as well assume $U = Y$. In order to make full use of (2.3), we need a special set-up, as follows: Assume (without loss of generality) that X and Y are retracts of locally finite complexes P and Q, respectively, where $\dim P \times I$ and $\dim Q$ are no greater than $\operatorname{sd}(X \times I, Y)$; let $r: P \to X$ and $s: Q \to Y$ be the retraction maps. Assume that both r and s are proper mappings, so that preimages of compact sets lie in finite subcomplexes. Finally, let d stand for the barycentric metric on P and Q, or any other metric under which closed, bounded sets are compact.

Let $\varepsilon: Y \to (0, \infty)$ be a continuous function. Applying (2.3) with $K = Q$, $L = \emptyset$, and $\varphi = s$, we get a proper mapping $v: Q \to X$ such that $d(fv, s) \leqq \varepsilon s$. Let $g = v \,|\, Y$. Then $g: Y \to X$ is a proper map such that

$$d(fg, id_Y) \leqq \varepsilon .$$

Clearly ε can be chosen so that fg is homotopic to the identity on Y via a homotopy $h: Y \times I \to Y$ with the properties

$$\begin{gathered} h \text{ is a proper homotopy ,} \\ h_0 = \text{identity on } Y \text{ and } h_1 = fg , \\ d(y, h_t(y)) \leqq 1 \text{ for } 0 \leqq t \leqq 1, y \in Y . \end{gathered}$$

Now, define $H: P \times I \to Y$ by

$$H(x, t) = h(fr(x), t), x \in P, t \in I .$$

Notice that H is a proper map: a sequence in $P \times I$ tends to infinity if and only if its image under H does so. Finally, define $\bar{h}: P \times \{0, 1\} \to X$ by

$$\bar{h}_0 = r, \bar{h}_1 = gfr .$$

Then $f\bar{h}_0 = fr = h_0 fr = H_0$, and $f\bar{h}_1 = fgfr = h_1 fr = H_1$; i.e.,

$$f\bar{h} = H | P \times \{0, 1\} .$$

Apply (2.3) again, with $K = P \times I$, $L = P \times \{0, 1\}$, $\varphi = H$, $\psi = \bar{h}$, and $\varepsilon = 1$. We get an extension $\bar{H}$ of $\bar{h}$ over $P \times I$, $\bar{H}: P \times I \to X$, such that

$$d(f\bar{H}, H) \leqq 1 .$$

That is, $d(f\bar{H}_t, H_t) \leqq 1$ for each t.

It is obvious that $\bar{H}$ is a homotopy between r and gfr, so that $\bar{H} | X \times I$ is a homotopy between the identity on X and gf. Moreover $\bar{H} | X \times I$ is a proper homotopy: If a sequence $\bar{H}(x_n, t_n)$ converges to a point $x_0 \in X$, then $f\bar{H}(x_n, t_n)$ converges to $f(x_0)$, so that $H(x_n, t_n)$ is bounded; since H is a proper map, (x_n, t_n) must have a convergent subsequence.

Before leaving this section, here is one final corollary to (2.3).

COROLLARY 2.4. *Let X be a locally compact metric space, let Y be an* ANR, *and let f be a proper UV^k−trivial map of X onto Y. Then, for each open subset U of Y,*

$$[f | f^{-1}(U)]_\#: \pi_q(f^{-1}(U)) \to \pi_q(U)$$

is an isomorphism for $0 \leqq q \leqq k$ and an epimorphism for $q = k + 1$.

Proof. We may as well assume that $U = Y$. That $f_\#$ is monic for $0 \leqq q \leqq k$ is obvious from (2.3). To see that $f_\#$ is epic for $0 \leqq q \leqq k + 1$ we need (2.3) together with the observation that "sufficiently close" maps into ANR's are homotopic. (Compare with Theorem 3.1.)

3. The image of a cell-like map. The cell-like image of an ENR is an ENR, provided that the image is finite-dimensional. This result is a corollary to the most general result of this section, Theorem 3.1.

THEOREM 3.1. *Let X and Y be locally compact metric spaces, and let f be a proper, UV^k-trivial mapping of X onto Y. If U is any open subset of Y then*

$$[f \mid f^{-1}(U)]_{\#}: \pi_q f^{-1}(U) \to \pi_q(U)$$

is an isomorphism for $0 \leqq q \leqq k$.

Before proving (3.1), we deduce two corollaries.

COROLLARY 3.2. *Let X and Y be locally compact metric spaces, with* $k = \dim Y < \infty$. *If there is a proper UV^k-trivial mapping of X onto Y, then Y is an* ENR.

This corollary follows immediately from (3.1) and the fact that an LC^k space Y is an ANR provided $k \geqq \dim Y < \infty$. See [13], § V.7.1. (Y is clearly locally compact. See the introductory lemma.) (The definition of LC^k many also be found in [13].) The following simplification is the main point:

COROLLARY 3.3. *Let X be an* ENR, *and let f be a proper, cell-like map of X onto Y. If Y is finite-dimensional and metrizable then Y is an* ENR.

The basic tool used in the proof of (3.1) is the following "homotopy" lemma. This lemma provides not only a "lifting to within ε-homotopy" theorem similar to (2.3), but also provides a continuous selection of liftings of approximations.

LEMMA 3.4. *Let the following be given*:
(i) *Locally compact metric spaces X and Y.*
(ii) *A pair (K, L) of finite simplicial complexes,* $\dim K \leqq k$.
(iii) *A proper, UV^k-trivial mapping f of X onto Y.*
(iv) *A map* $\varphi: K \to Y$.
(v) *A map $\psi: L \to X$ such that* $f\psi = \varphi \mid L$.
Then there exist maps $\Phi: K \times I \to Y$ and $\Psi: K \times (0, 1] \to X$ such that
(1) $f\Psi_t = \Phi_t$ *for* $0 < t \leqq 1$,
(2) $\Psi_t \mid L = \psi$ $o < t \leqq 1$, *and*
(3) $\Phi_0 = \varphi$.

Proof. For each $y \in Y$ let $\{U_y^{(n)}\}$ be a sequence of neighborhoods of y such that

$$\operatorname{diam} U_y^{(n)} \leqq \frac{1}{n}$$

and

$$\bar{U}_y^{(n+1)} \subset U_y^{(n)} .$$

Moreover, construct the $U_y^{(n)}$ so that any singular q-sphere in $f^{-1}(U_y^{(n+1)})$ bounds a singular $(q+1)$-disk in $f^{-1}(U_y^{(n)})$. Finally, construct $U_y^{(1)}$ to have compact closure.

Using an argument similar to that of (2.3), or in fact applying (2.3) carefully, we can find a sequence $\{\psi_n\}$ of maps of K into X such that

$$\psi_n | L = \psi ,$$

and

$$f\psi_n(x) \in U_{\varphi(x)}^{(n+1)}$$

for each n and each $x \in K$. Being slightly more careful, we can find a descending sequence $K_1, K_2, \cdots$ of subdivisions of K such that

$$f\psi_n(\sigma) \subset U_{\varphi(x)}^{(n+1)} \text{ for all } x \in \sigma$$

holds for each n and each $\sigma \in K_n$.

SUBLEMMA. *For each n there is a map $\Psi^{(n)}: K \times I \to X$ such that*

$$\Psi_0^{(n)} = \psi_{(k+1)(n+1)} ,$$
$$\Psi_1^{(n)} = \psi_{(k+1)n} ,$$

and

$$f\Psi^{(n)}(x \times I) \subset U_{\varphi(x)}^{(k+1)n} .$$

Proof of sublemma. Extend $\Psi_0 \cup \Psi_1$ over the cells of $K_{(k+1)(n+1)} \times I$ as follows. Let $J = K_{(k+1)(n+1)}$. Use the triviality of the inclusion $f^{-1}(U_{\varphi(x)}^{(k+1)(n+1)}) \subset f^{-1}(U_{\varphi(x)}^{(k+1)(n+1)-1})$ to extend over the cells of $J^0 \times I$, where $x \in J^0$. Then, using the triviality of the inclusions

$$f^{-1}(U_{\varphi(x)}^{j+1}) \subset f^{-1}(U_{\varphi(x)}^{j})$$

for each j and each $x \in \sigma \in J$, extend over the cells of $J^p \times I$ for each $p \leqq k$.

Now, let $\Psi: K \times (0, 1] \to X$ be the composition of the $\Psi^{(n)}$, where $\Psi^{(n)}$ is to be copied on the interval $[1/(n+1), 1/n]$. Let $\Phi: K \times I \to Y$ be defined by

$$\Phi(x, t) = \begin{cases} f\Psi(x, t) & \text{if } 0 < t \leqq 1 \\ \varphi(x) & \text{if } t = 0 . \end{cases}$$

Clearly Φ is a well-defined function. Φ is continuous, since Φ_t converges uniformly to φ as $t \to 0$.

Proof of (3.1). We may as well assume that $U = Y$. Let $\varphi: S^q \to Y$ be a map, $q \leqq k$. Then, by (3.4), there is a map $\bar{\varphi}: S^q \to X$ such that $f\bar{\varphi}$ is homotopic to φ. Hence $f_\#[\bar{\varphi}] = [\varphi]$, and $f_\#$ is epic for $0 \leqq q \leqq k$.

Now, suppose that $\psi_0, \psi_1: S^q \to X$ and $f\psi_0$ is homotopic to $f\psi_1$. Then, using (2.3), we can "lift" the homotopy, provided $q \leqq k$. Hence $f_\#$ is monic for $0 \leqq q \leqq k$, and the proof is complete.

Note, incidentally, that due to the relative nature of (2.3) and (3.4), we need not worry about base points.

4. Cell-like maps defined on manifolds. Any topological manifold (with or without boundary) is an ENR. (See [12].) Hence our results for cell-like maps of ENR's hold *a fortiori* for cell-like maps of manifolds. Recall our conventions about manifolds: Unless specifically stated, manifolds, *are not* assumed to be compact *nor* are they assume equipped with any extra structure; however, manifolds *are* assumed to have empty boundary.

One question we would like to consider in this section is: (1) When is a cell-like map on a manifold actually cellular? (A *cellular map* on a manifold M is one whose point-inverses are cellular in M.) Another way to state this question is: (1)′ If $f: M \to Y$ is a cell-like map, and if M is a manifold, when is it true that $f^{-1}(y)$ is cellular in M for each $y \in Y$? This reformulation immediately poses: (2) If $f: M \to N$ is a cell-like map of manifolds, and if C is cellular in N, is $f^{-1}(C)$ cellular in M? These questions are answered, at least partially, in this section.

Homotopy-manifolds. A *homotopy-m-manifold* is an ENR M such that for any point x of M, x has arbitrarily small neighborhoods $V \subset U$ in M, with $\bar{V} \subset U$, and with the property: The image of $\pi_q(V - x)$ in $\pi_q(U - x)$ (under the map induced by inclusion) is isomorphic to $\pi_q(S^{m-1})$ for $q \geqq 0$. (Compare with [9] and [10].)

THEOREM 4.1. *Let M be an m-manifold, and let $f: M \to Y$ be a proper cellular map of M onto the finite-dimensional metric space Y. Then Y is a homotopy-m-manifold.*

Proof. (Compare with [18].) f is cell-like, so that Y is an ENR by (3.3). For $y \in Y$ and neighborhoods $V \subset U$ of y in Y, consider the following commutative diagram

$$\begin{array}{ccc} \pi_q(f^{-1}(V - y)) & \longrightarrow & \pi_q(f^{-1}(U - y)) \\ \downarrow & & \downarrow \\ \pi_q(V - y) & \longrightarrow & \pi_q(U - y)\,. \end{array}$$

The vertical arrows are induced by $f|$, and hence are isomorphisms by (1.2). The horizontal arrows are induced by inclusions. Thus the problem of showing that Y is a homotopy m-manifold is transferred to a similar problem in M.

Using cellularity of $f^{-1}(y)$, let W_3 be an open m-cell in M containing $f^{-1}(y)$. Let U be a neighborhood of y such that $f^{-1}(U) \subset W_3$. Now, $W_3 - f^{-1}(y) \approx S^{m-1} \times R^1$. (See [5].) Thus there is an open m-cell W_2, containing $f^{-1}(y)$ and lying in $f^{-1}(U)$, such that the inclusion

$$(W_2 - f^{-1}(y)) \subset (W_3 - f^{-1}(y))$$

is a homotopy equivalence. Let V be a neighborhood of y such that $f^{-1}(V) \subset W_2$, and find an open m-cell W_1 such that $f^{-1}(y) \subset W_1 \subset f^{-1}(V)$ and the inclusion $(W_1 - f^{-1}(y)) \subset (W_2 - f^{-1}(y))$ is a homotopy equivalence. We have the following commutative diagram (in which all maps are induced by inclusions):

$$\begin{tikzcd}
\pi_q(W_1 - f^{-1}(y)) \arrow[rr, "\approx"] \arrow[dr] & & \pi_q(W_2 - f^{-1}(y)) \arrow[rr, "\approx"] \arrow[dr, "\gamma"] & & \pi_q(W_3 - f^{-1}(y)) \, . \\
& \pi_q(f^{-1}(V - y)) \arrow[rr, "\alpha"] \arrow[ur, "\beta"] & & \pi_q(f^{-1}(U - y)) \arrow[ur] &
\end{tikzcd}$$

The fact that the upper horizontal arrows are isomorphisms implies that β is epic and γ is monic. Therefore

$$\operatorname{Im} \alpha = \operatorname{Im} \gamma \approx \pi_q(W_2 - f^{-1}(y)) \approx \pi_q(S^{m-1}) \ ,$$

and the proof is complete.

A piecewise linear, or *PL*, manifold is a manifold which has a *PL* structure, called a polystructure in [23]. The next theorem gives a complete answer to question (1) in the case where M is a *PL* manifold of dimension at least five. First, a definition.

Property SUV^k. An embedding $\varphi: A \to X$ has Property SUV^k ("S" is for "strong") if for each open set U of X containing $\varphi(A)$ there is an open set V of x, with $\varphi(A) \subset V \subset U$, such that any map $S^q \to (V - \varphi(A))$ can be extended to a map $B^{q+1} \to (U - \varphi(A))$, $0 \leqq q \leqq k$.

SUV^k-trivial maps. A map $f: X \to Y$ is SUV^k-trivial if, for each $y \in Y$, the inclusion $f^{-1}(y) \subset X$ has Property SUV^k.

REMARK. Property SUV^1 is what was called "Property (*)" in [19]. McMillan [22] immortalized Property SUV^1 and Property UV^∞ when he proved that an embedding $\varphi: A \to M$ has cellular image, provided M is a PL manifold of dimension at least five and φ has Properties SUV^1 and UV^∞. The following theorem provides a converse to McMillan's criterion for upper-semicontinuous families.

THEOREM 4.2. *Let M be a PL m-manifold, $m \geqq 5$, and let $f: M \to Y$ be a proper, cell-like map of M onto the finite-dimensional metric space Y. Then the following conditions are equivalent*:

(a) *f is cellular.*
(b) *f is SUV^1-trivial.*
(c) *f is SUV^k-trivial for $k \leqq m - 2$.*
(d) *Y is a homotopy-m-mainfold.*

Proof. (a) $\Rightarrow$ (d) is a special case of (4.1). (d) $\Rightarrow$ (c) follows immediately from (3.3) and (1.2) (c). (c) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (a) follows immediately from McMillan's cellularity criterion as interpreted in the above remark.

Recall that an *open n-cell* is a manifold homeomorphic to R^n.

THEOREM 4.3. *Let M and N be topological manifolds,* $\dim N = n \geqq 5$, *and let $f: M \to N$ be a proper cell-like mapping.*

(1) *Let U be an open subset of N. Then U is an open n-cell if and only if $f^{-1}(U)$ is an open n-cell.*

(2) *If C is a cellular subset of N then $f^{-1}(C)$ is cellular in M. In particular,*

(3) *f is a cellular map.*

Proof. It is easy to see that (1) $\Rightarrow$ (2), since any compact subset of an open cell V lies interior to a closed cell in V. Also, (2) $\Rightarrow$ (3) trivially, so we need only prove (1).

But (1) follows immediately from a recent result of Siebenmann [26]. He shows that an open topological manifold which is properly homotopically equivalent to R^n must be homeomorphic to R^n, provided $n \geqq 5$. $(f \mid f^{-1}(U): f^{-1}(U) \to U$ is a proper homotopy equivalence by Theorem 1.2.)

COROLLARY 4.4. *If $f: M \to N$ is a proper, cell-like map of topological manifolds of dimension at least five, then $M \approx R^n$ if and only if $N \approx R^n$.*

We conclude with the analogue of (4.3) for manifolds with boundary. An *open n-half-cell* is a manifold-with-boundary homeomorphic to $R^{n-1} \times [0, \infty)$.

THEOREM 4.5. *Let M and N be topological manifolds with* *boundary,* $\dim N = n \geqq 6$, *and let* $f: M \to N$ *be a proper, cell-like map such that* $f(\mathrm{Bd}\, M) = \mathrm{Bd}\, N$ *and* $f(\mathrm{Int}\, M) = \mathrm{Int}\, N$.

(1) *Let* U *be an open subset of* N. *Then* U *is an open* n*-half-cell if and only if* $f^{-1}(U)$ *is an open* n*-half-cell.*

(2) *If* C *is cellular-at-the-boundary of* N *then* $f^{-1}(C)$ *is cellular-at-the-boundary of* M.

Proof. Again it is clear that (1) $\Rightarrow$ (2). To prove (1), let U be an open subset of N, $V = f^{-1}(U)$. Then $\mathrm{Bd}\, U = U \cap \mathrm{Bd}\, N$ and $\mathrm{Bd}\, V = V \cap \mathrm{Bd} M$, so that $\mathrm{Bd}\, V = f^{-1}(\mathrm{Bd}\, U)$ and $\mathrm{Int}\, V = f^{-1}(\mathrm{Int}\, U)$. Also, $f \mid \mathrm{Bd} M$ and $f \mid \mathrm{Int} M$ are cell-like maps. Applying (4.4), we see that $\mathrm{Bd}\, V \approx R^{n-1}$ if and only if $\mathrm{Bd}\, U \approx R^{n-1}$ and that $\mathrm{Int}\, V \approx R^n$ if and only if $\mathrm{Int}\, U \approx R^n$. It is known that a topological m-manifold-with-boundary, say W, is an open m-half-cell provided $\mathrm{Bd}\, W \approx R^{m-1}$, $\mathrm{Int}\, W \approx R^m$, and $m \geqq 4$. See [6] and [11]. The result now follows.

S. Armentrout, T. Price, and G. Kozlowski have independently discovered some of these results, working from entirely different points, of view. See [1] and [17]. In particular, both Price and Kozlowski have versions of (2.3), Kozlowski has a version of (3.4), and Armentrout has studied property UV^k. Finally, A. V. Cernavskii has informed me that he and V. Kompaniec have obtained some results related to these, although not as general. See [14].

The referee has pointed out that the arguments given for Lemmas 2.3 and 3.4 are "embellishments" of arguments given by Price in [24]. (In fact, Price's arguments are similar to some of the arguments given by Smale in [27], which is where some of the ideas in the present paper originated.) See [21] and [23] for further discussions along this line.

The terminology introduced in [19] and [20], Property (**), has been changed so that the present paper is now in agreement with at least *part* of the existing literature.

A property equivalent to "cell-like" (for finite dimensional compacta) is studied by Hyman in [15] under the name "absolute neighborhood contractibility". (This is the property described in condition (d) of Theorem 1.1.) Some of the arguments of [19] are quite similar to some in [15], as are some of the results. Hyman's result that an absolutely neighborhood contractible space is an ANR divisor translates, using Theorem 1.1 and the terminology above, as follows. If X is an ANR, and if A is a cell-like subset of X, then X/A is an ANR. (Compare this with Corollary 3.3, but note that Hyman uses a more general definition of ANR.)

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INSTITUTE FOR ADVANCED STUDY AND
FLORIDA STATE UNIVERSITY