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A MONOTONICITY PRINCIPLE FOR EIGENVALUES

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The smallest eigenvalue of certain boundary problems for second order linear elliptic partial differential equations increases to infinity as the domain in question shrinks to the empty set. The object of this note is to formulate and prove an analogous result for linear elliptic differential operators L of general even order. Specifically, let $G(t)$ be a bounded domain in n-dimensional Euclidean space, and suppose that $G(t)$ has thickness t (in a sense which will be precisely defined below). Let $\lambda_0(t)$ be the smallest eigenvalue of a boundary problem associated with L and $G(t)$. It will be shown that $\lambda_0(t)$ increases to infinity as t tends to zero from the right.

The proof depends on a generalization of Agmon's form [1] of Poincaré's inequality. In the second-order case, a monotonicity principle of the type under consideration has been applied to obtain oscillation theorems (cf. [3], [4]) for partial differential equations on unbounded domains.

2. Preliminary lemmas. Let G be a domain (not necessarily bounded) in n-dimensional Euclidean space R^n. We shall say that G has *bounded thickness* $\leqq s$, or simply *thickness* $\leqq s$, if and only if there is a line ℓ such that each line parallel to ℓ intersects G in a set each of whose components (i.e., maximal connected subsets) has diameter $\leqq s$. For example, if $|x|$ denotes the length $(\sum x_i^2)^{1/2}$ of the vector $x = (x_1, \cdots, x_n)$ in R^n, then the annulus $\{x \in R^n : r_0 < |x| < r_1\}$, $r_0 > 0$, has thickness $\leqq 2\sqrt{[r_1^2 - r_0^2]}$.

Let $C^m(G)$ denote the class of all m times continuously differentiable real-valued functions on G, and $C_0^m(G)$ denote the class of all C^m functions having compact support in G. We use the standard multi-index notation: let $\alpha = (\alpha_1 \cdots, \alpha_n)$ have nonnegative integral components and "norm" $|\alpha| = \alpha_1 + \cdots + \alpha_n$; let $D_i^{\alpha_i}$ denote the partial differential operator $(\partial^{\alpha_i}/\partial x_i^{\alpha_i})$, and let $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$.

LEMMA 1. *If G has bounded thickness $\leqq s$ and if every line parallel to the line ℓ in the definition of bounded thickness intersects G in a set with at most k components, where k is some positive integer, then*

$$|v|_{j,G} \leqq (ks)^{m-j} |v|_{m,G} \tag{1}$$

for all $v \in C_0^m(G)$, $0 \leqq j \leqq m-1$, where

$$| v |_{j,G} = \left| \int_G \sum_{|\alpha|=j} | D^\alpha v |^2 dx \right|^{1/2} .$$

Proof. We refine the argument given in [1, pp. 74–75]. Let ℓ' be a line parallel to ℓ, and assume that x^0 and $x^0 + q$ are points in $\ell' \cap \partial G$ such that $\ell' \cap G$ is contained in the segment between x^0 and $x^0 + q$. By defining v to vanish outside G, we can assume that $v \in C_0^m(R^n)$. For $-\infty < t < +\infty$ let $f(t) = v(x^0 + t | q |^{-1} q)$. Then $f(0) = 0$, so that

$$f(t) = \int_0^t f'(r) dr .$$

Since v vanishes outside G,

$$f(t) = \int_{K(G,t)} f'(r) dr ,$$

where

$$K(G, t) = \{r \colon r \leqq t \text{ and } x^0 + r | q |^{-1} q \in \ell' \cap G\} .$$

This set is by hypothesis a union of at most k disjoint intervals, the sum of whose lengths is at most ks. By Schwarz's inequality,

$$| f(t) |^2 \leqq ks \int_{K(G,t)} | f'(r) |^2 dr \leqq ks \int_{-\infty}^{\infty} | f'(r) |^2 dr .$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} | f(t) |^2 dt &= \int_{K(G,|q|)} | f(t) |^2 dt \\ &\leqq (ks)^2 \int_{-\infty}^{\infty} | f'(r) |^2 dr . \end{aligned}$$

Now express $| v |_{0,G}^2$ as an iterated integral with one of the integrations taken in the direction of ℓ. From the last inequality above it follows that

$$| v |_{0,G}^2 \leqq (ks)^2 | v |_{1,G}^2 .$$

Applying this inequality to $D_i v$, we obtain

$$| D_i v |_{0,G}^2 \leqq (ks)^2 | D_i v |_{1,G}^2 .$$

Summing over all i, we obtain

$$| v |_{1,G}^2 \leqq (ks)^2 | v |_{2,G}^2 .$$

The conclusion of the lemma now follows by induction.

In the application mentioned in the introduction, if G is an annulus with $r_1 - r_0 = t$, it is important to have an inequality of the form

$$|v|_{0,G} \leqq g(t)\,|v|_{m,G}\,,$$

where the function g is monotone strictly increasing. Such an inequality follows immediately from Lemma 1, but does not appear to be readily obtainable from the corresponding result in [1].

We now consider the $2m$-th order linear elliptic partial differential operator L defined by

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(A_{\alpha\beta}D^\beta u) + Bu\,, \tag{2}$$

where m is a positive integer, and $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ are multi-indices with nonnegative integral components. The coefficients $A_{\alpha\beta}$ are supposed to be real-valued, symmetric in the indices, and have bounded continuous derivatives of all orders $\leqq m$ on G. The coefficient B is real-valued, bounded, and continuous on G. For each $z \in R^n$ we write $z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}$.

Let $p(m)$ denote the number of distinct multi-indices α satisfying $|\alpha| = m$. For operators of the kind defined by (2), we shall suppose that there exists a number $E > 0$ such that for all $x \in G$ and all $p(m)$-tuples $\{\xi_\alpha : |\alpha| = m\}$ of real numbers ξ_α

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geqq E \sum_{|\alpha|=m} \xi_\alpha^2\,. \tag{3}$$

Without loss of generality we assume that we may take $\xi_\alpha = \xi_\gamma$ if $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a permutation of $\alpha = (\alpha_1, \cdots, \alpha_n)$. We note that the usual ellipticity condition is

(a) The form $\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)z^{\alpha+\beta}$ is positive definite at each point $x \in G$.

If the coefficients $A_{\alpha\beta}$ are constant or if Lu has the form

$$(-1)^m \sum_{i,j=1}^{n} D_i^m(a_{ij}D_j^m u) + Bu\,,$$

it can be shown that (3) is a consequence of the ellipticity condition (a). We now define the quadratic functional

$$J[u] = \int_G \sum_{|\alpha|=|\beta|=m} (A_{\alpha\beta}D^\alpha u D^\beta u + Bu^2)dx$$

for $u \in C_0^m(G)$. Then the following special case of Gording's inequality is valid.

LEMMA 2. *Let* $A_{\alpha\beta}$ *satisfy* (3). *Then there exists a number* $b > -\infty$ *such that*

$$J[u] \geqq E\,|\,u\,|^2_{m,G} + b\,|\,u\,|^2_{0,G} \tag{4}$$

for all $u \in C_0^m(G)$.

Proof. Since B is bounded and continuous on G, there exists a number $b > -\infty$ such that

$$\int_G Bu^2 dx \geqq b \int_G u^2 dx \tag{5}$$

for all $u \in C_0^m(G)$. Condition (3) yields

$$\int_G \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta} D^\alpha u D^\beta u\, dx \geqq E \int_G \sum_{|\alpha|=m} (D^\alpha u)^2\, dx\ .$$

Combining this with inequality (5) we obtain (4), and the lemma is proved.

Our next preliminary result is a form of Courant's variational principle [2].

LEMMA 3. *Let G be a bounded domain, with boundary ∂G having a piecewise continuous unit normal. The function $u_0 \in C^{2m}(G)$ which minimizes the functional $J[u]$ under the condition $|\,u\,|_{0,G} = 1$ is an eigenfunction corresponding to the smallest eigenvalue of the problem*

$$Lu = \lambda u \text{ in } G,\ D^\alpha u = 0 \text{ on } \partial G,\ 0 \leqq |\,\alpha\,| \leqq m - 1\ . \tag{6}$$

Proof. According to [5, §§ 11, 28], there exists a minimizing function u_0 which is a weak solution of (6) in the following sense:

$$\langle u_0, (L - \lambda_0)v\rangle = 0\ , \qquad v \in C_0^\infty(G)\ ,$$

where $\langle\, , \rangle$ is the usual $L^2[G]$ inner product and λ_0 is the minimum value of $J[u]$. The results of [1, §§ 8, 9] now imply that $u_0 \in C^{2m}(G)$ and u_0 satisfies $Lu = \lambda_0 u$. A standard argument [2, p. 400] now shows that λ_0 is the smallest eigenvalue of (6).

3. The main result.

For $0 < t < \infty$ let G_t be a bounded domain having a piecewise smooth boundary ∂G_t. We suppose that G_t has thickness $\leqq t$, and that the line ℓ in the definition of bounded thickness intersects G_t in a set with at most k components. We suppose that $A_{\alpha\beta} \in C^m(\bar{G}_t)$ and that B is continuous on $\bar{G}_t$. We also suppose that the coefficients $A_{\alpha\beta}$ satisfy condition (3) on G_t.

THEOREM. *If $0 < r < s < \infty$ implies that G_r is a proper subset of G_s, then the smallest eigenvalue $\lambda_0(t)$ of the boundary problem*

$$Lu = \lambda u \text{ in } G_t;\ D^\alpha u = 0 \text{ on } \partial G_t,\ 0 \leqq |\alpha| \leqq m - 1$$

is monotone nonincreasing in t, and $\lim_{t\to 0+} \lambda_0(t) = +\infty$.

Proof. Introduce the notation

$$J_t[u] = \int_{G_t} \sum_{|\alpha|=|\beta|=m} (A_{\alpha\beta} D^\alpha u D^\beta u + Bu^2) dx ,$$

and

$$\| u \|_t = | u |_{0,G_t},\ | u |_{m,t} = | u |_{m,G_t} .$$

By Lemma 3,

$$\lambda_0(t) = \inf \{J_t[u]/\| u \|_t^2 \colon u \in C^{2m}(G_t)\} .$$

Since G_t increases with t, it is clear that the class of admissible functions is nondecreasing, and therefore $\lambda_0(t)$ is nonincreasing in t. By Lemma 2, there exist numbers $E(t) > 0$, $b(t) > -\infty$ such that

$$J_t[u] \geqq E(t) | u |_{m,t}^2 + b(t) \| u \|_t^2 . \tag{7}$$

According to Lemma 1,

$$\| u \|_t^2 \leqq (kt)^{2m} | u |_{m,t}^2 .$$

Combining this with inequality (7) we obtain

$$J_t[u] \geqq [(kt)^{-2m} E(t) + b(t)] \| u \|_t^2 .$$

Hence

$$\lambda_0(t) \geqq (kt)^{-2m} E(t) + b(t) .$$

Since $E(t)$ may be chosen to be the infimum of

$$\Big(\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x) \xi_\alpha \xi_\beta\Big) \Big/ \Big(\sum_{|\alpha|=m} \xi_\alpha^2\Big)$$

over all $x \in G_t$ and all $p(m)$-tuples $\{\xi_\alpha \colon |\alpha| = m\}$ of real numbers, it is clear that $E(t)$ cannot decrease as t decreases, so that $\liminf_{t\to 0+} E(t) > 0$. Moreover, since B is bounded and continuous on $\bar{G}_t$, there exists r such that $\liminf_{t\to 0+} b(t) > r > -\infty$. Hence $\lim_{t\to 0+} \lambda_0(t) = +\infty$.

The author (Ph. D. Thesis, University of British Columbia) has applied a form of this theorem (in the cases where G_t is an annulus or a finite cylinder in R^n) in the derivation of oscillation theorems for elliptic differential equations of even order $2m$.

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