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RINGS IN WHICH MINIMAL LEFT IDEALS ARE PROJECTIVE

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Let R be an associative ring with identity. Then the left socle of R is a direct summand of R as a right R-module if and only if it is projective as a left R-module and contains no infinite sets of orthogonal idempotents. This implies, for example, that a ring with finitely generated left socle and no nilpotent minimal left ideals is a ring direct sum of a semisimple artinian ring and a ring with zero left socle.

A ring with projective, essential, finitely generated left socle has maximal and minimal condition on annihilater left and right ideals. A left or right perfect ring satisfying these hypotheses is semiprimary. However, there are nonsemiprimary left perfect rings with projective, finitely generated, nonzero left socle.

This paper has in part grown out of another paper in which the author will characterize semiperfect rings with projective, essential socle. Section 2 arose from attempts to find a simple proof that such rings are ring direct sums of indecomposable rings sharing the same properties (this is true). In Theorem 2.1, we show that any ring in which the identity is a sum of primitive idempotents is a (necessarily unique and finite) ring direct sum of indecomposable rings. Thus a ring has projective (projective essential) socle and no infinite sets of orthogonal idempotents if and only if it is a ring direct sum of indecomposable rings satisfying the same hypotheses.

Theorem 3.1 gives a list of conditions which are equivalent to the existence of a projective, essential left socle in an arbitrary ring. With these equivalent definitions in mind, the reader may note that Colby and Rutter [3, Th. 2.9] and Gordon [4, Th. 4.1] have characterized left artinian rings R with projective (trivially essential) left socle in which every indecomposable direct summand of ${}_RR$ has a unique simple submodule. In a recent paper, Zaks has extended this result to semiprimary rings [7, Th. 1.4]. But in Theorem 3.4 we show that a left (or right) perfect ring with finitely generated, projective and essential left socle is semiprimary. So Zaks' characterization is automatically pushed up to perfect rings-with one defect. There exist left perfect rings with projective, nonessential left socle for which every indecomposable left module direct summand of the ring contains a unique simple (Example 4.4). (Such a ring obviously fails to be semiprimary.) We cannot handle this situation.

1. The case where the socle is projective. All rings in this paper are assumed to have an identity. Furthermore, if $f: R \rightarrow R'$ is a ring homomorphism, then f maps the identity of R to the identity of R'. Thus, we are assuming in particular that all modules are unitary.

Recall that if an R-module is a direct sum of a family of submodules, then it is projective if and only if every submodule in the family is projective. We shall make repeated use of this fact. For instance, if M is an R-module, then the socle of M is projective if and only if every simple submodule of M is projective.

Finally, an ideal is a two-sided ideal. We reserve the letter J for the (Jacobson) radical and write $J(R)$ when the context is not clear.

LEMMA 1.1. *Suppose R is a ring with projective left socle S. If T is a sum of homogeneous components*[1] *of S, then $T \cap I = TI$ for any left ideal I. In particular, the socle of a left ideal K is SK.*

Proof. According to Gordon [4, Lemma 1.3], it is sufficient to show that T intersects its right annihilator trivially. Since $T \cap T^r$ is a completely reducible left R-module, we need only show that no simple submodule of T right annihilates T. Let Q be such a simple submodule. Since Q is projective, $Q \simeq \mathrm{Re}$, $e^2 = e$. But Q and Re must both belong to the same homogeneous component of the socle. In particular, $Re \subseteq T$. But $Re\,Q \neq 0$ (since Re and Q have same left annihilator in R). Therefore $TQ \neq 0$.

THEOREM 1.2. *In any ring R, the following conditions are equivalent.*

(1) *The left socle of R is a projective left R-module and contains no infinite sets of orthogonal idempotents*[2].

(2) *The left socle of R is a direct summand of R as a right R-module.*

Futhermore, if either of these conditions hold, then R has only finitely many isomorphism classes of minimal left ideals.

Proof. (1) implies (2). Let S be the left socle of R. Since S has no infinite sets of orthogonal idempotents, S must contain a maximal set $\{e_1, \cdots, e_n\}$ of orthogonal idempotents. Then

$$S = Re \oplus S \cap R(1 - e)$$

where $e = e_1 + \cdots + e_n$. Let f be an idempotent in $S \cap R(1 - e)$ and

[1] See Jacobson [5, p. 63] for the definition.

[2] There are well known examples of semiprime rings in which the socle is not finitely generated and thus contains an infinite set of orthogonal idempotents.

set $g = (1 - e)f$. Then g is an idempotent in S orthogonal to every e_i. The choice of the set $\{e_1, \cdots, e_n\}$ implies $g = 0$. Thus $f = f^2 = f(ef) = (fe)f = 0$. It follows that $S \cap R(1 - e) \subseteqq J$ and hence that Re is isomorphic to $S + J/J$ as left R-modules. So the hypothesis in (1) implies that every minimal left ideal embeds in Re. Since $S \cap R(1 - e) \subseteqq J$, we have

$$S \cap [R(1 - e)Re] = SR(1 - e)Re = [S \cap R(1 - e)]Re = 0$$

by Lemma 1.1. Thus $(1 - e)Re = 0$. This implies that e acts as a left identity on every projective minimal left ideal. Since all minimal left ideals are projective by hypothesis, $S = eR$ follows.

(2) implies (1). We have $S = fR$, $f^2 = f$. Since Rf is both projective and completely reducible, every homomorphic image of Rf is isomorphic to a direct summand of Rf and thus projective. But, since f acts as a left identity on S, every minimal left ideal is clearly an image of Rf. Hence ${}_RS$ is projective. If $g = g^2 \in S \cap R(1 - f)$, then $g = g(fg) = (gf)g = 0$. So $S \cap R(1 - f) \subseteqq J$. Thus the existence of an infinite set of orthogonal idempotents in S would contradict the composition series length of $S + J/J$ (which is just the length of Rf).

It is obvious from the proof of (1) $\langle = \rangle$ (2) that S has no infinite sets of orthogonal idempotents if and only if S is finitely generated modulo J. This implies both the last statement of the theorem and the following corollary.

COROLLARY. *If the socle of R/J is finitely generated, then the left socle of R is projective as a left R-module if and only if it is a direct summand of R as a right R-module.*

2. Primitive idempotents and related subjects. By a primitive idempotent, we mean a nonzero idempotent which cannot be written as a sum of two nonzero orthogonal idempotents. We say that primitive idempotents e and f in a ring R are *linked* if there exist finite sequences $e = e_0, e_1, \cdots, e_n = f$ and $f_1, f_2, \cdots, f_n$ of primitive idempotents of R such that f_iRe_{i-1} and f_iRe_i are both nonzero for $1 \leqq i \leqq n$. Linking is obviously an equivalence relation on the set of all primitive idempotents in R. By a *block* of R we mean the sum of all principal left ideals in R which are generated by primitive idempotents belonging to the same equivalence class.

The following is a generalization of a theorem which appears in Curtis and Reiner [3, p. 378, Th. 55.2].

THEOREM 2.1. *Let R be a ring in which the identity is a sum of primitive idempotents. Then R is a unique ring direct sum (necessarily finite) of indecomposable rings. In fact, the indecom-*

posable ring direct summands of R are just the blocks of R.

Proof. According to [5, p. 42, Th. 1], it is enough to prove the existence of such a decomposition.

The hypothesis of the theorem implies that R is the sum of its blocks. Let B and B' be distinct blocks. If e and e' are primitive idempotents in B and B', respectively, then eRe' must be zero. Otherwise, we would have $eRe \neq 0$ and $eRe' \neq 0$ against the assumption that e and e' belong to different blocks and are therefore not linked. Hence $BB' = 0$. Then, since any block B obviously satisfies $B^2 \subseteqq B$ and R has an identity, it follows that R is a ring direct sum of its blocks. In particular, the number of distinct blocks must be finite (since $1 \in R$).

To finish the proof, we must show that an arbitrary block B is indecomposable. So, write $B = P \oplus Q$ where P and Q are ideals in B (and thus in R). We show first that any primitive idempotent in B belongs to either P or Q: Let $e \in B$ be a primitive idempotent. Since B is a ring direct summand of R, $Re = Be = Pe \oplus Qe$. But, e is primitive, so Re is indecomposable as a left R-module. Hence $Pe = 0$ or $Qe = 0$, so $e \in P$ or $e \in Q$ as was claimed.

Now let e, f, g be primitive idempotents in B such that $gRe \neq 0$ and $gRf \neq 0$. Clearly, $gRe \subseteqq gR \cap Re$ and $gRf \subseteqq gR \cap Rf$ so that $gR \cap Re \neq 0$ and $gR \cap Rf \neq 0$. Then, if $e \in P$, we must have $g \in P$ which forces $f \in P$. So, by induction, no primitive idempotent in P can be linked to one in Q and conversely. Since either P or Q contains a primitive idempotent, B must be indecomposable.

We would like to give a nicer characterization of linking in general rings. This seems to be very difficult. We do make a tenuous attempt:

LEMMA 2.2. *Let M be an R-module and P a finitely generated, projective R-module. Then* $\mathrm{Hom}_R(P, M) \neq 0$ *if and only if there exists a submodule K of M such that* $\mathrm{Hom}_R(P/JP, M/K) \neq 0$ *(here* $J = J(R)$*).*

Proof. Suppose $f: P \to M$ is nonzero and consider the composite map

$$P \xrightarrow{f} M \xrightarrow{nat} M/J \operatorname{im} f .$$

If this map were zero, we would have $\operatorname{im} f \subseteqq J \operatorname{im} f$. But $\operatorname{im} f$, as an image of a finitely generated module, is finitely generated. Since $J \operatorname{im} f = \operatorname{im} f$, Nakayama's Lemma implies $\operatorname{im} f = 0$ against $f \neq 0$. Therefore, since the composite map above is obviously zero on JP, it induces a nonzero homomorphism from P/JP into $M/J\,\mathrm{im}\,f$.

For the converse, suppose that for some submodule K of M, we have a nonzero map $P/JP \to M/K$. Thus we have a nonzero map $P \to M/K$. Since P is projective, this last map lifts to a nonzero map from P into M.

REMARK. If $\bigcap_{i=1}^{\infty} J^i M = 0$, the proof of 2.2 shows that a nonzero homomorphism from P into M induces a nonzero homomorphism from P/JP into $J^i M/J^{i+1}M$ for some i.

We need the following folk lemma.

PROPOSITION 2.3. *If e is an idempotent in a ring R, then the following statements are equivalent.*

(1) *e is a local idempotent*[3].

(2) *Re/Je is simple.*

(3) *Re has a unique maximal submodule.*

Proof. (1) $\Longleftrightarrow$ (2). For $x \in R$, denote the canonical image of x in R/J by $\bar{x}$. Then $\bar{e}\bar{R}\bar{e} \simeq eRe/eJe$ as rings and $\bar{R}\bar{e} \simeq Re/Je$ both as R-modules and as R/J-modules. But $\bar{R}$ is a semiprime ring. Therefore, $\bar{R}\bar{e}$ is a minimal left ideal in $\bar{R}$ if and only if $\bar{e}\bar{R}\bar{e}$ is a division ring [5, p. 65, Proposition]. Thus we need only recall that the radical of eRe is eJe.

(2) $\Longleftrightarrow$ (3). Since Re is projective, $Je = JRe$ is the intersection of the maximal submodules of Re [1, p. 474].

PROPOSITION 2.4. *Let e and f be primitive idempotents in a ring R and consider the condition:* (C) *There exists a finite sequence $e = e_0, e_1, \cdots, e_n = f$ of primitive idempotents of R with the property that some factor module of Re_{i-1} has a simple submodule isomorphic to a simple submodule of a factor module of Re_i for $1 \leqq i \leqq n$.*

Then the following statements hold.

(1) *If every primitive idempotent in R is local, then* (C) *is a necessary condition for e and f to be linked.*

(2) *If R/J is artinian, then* (C) *is sufficient condition for e and f to be linked.*

Proof. We remark that if $e^2 = e \in R$ and M is a left R-module, then

$$\mathrm{Hom}_R(Re, M) \simeq eM$$

[3] A ring is local if it is a division ring modulo its radical. So, by a local idempotent, we mean an idempotent e in a ring R with the property that eRe is a local ring.

as left eRe-modules. In particular, $\mathrm{Hom}_R(Re, M) \neq 0$ if and only if $eM \neq 0$.

(1) Assume every primitive idempotent of R is local and that e and f are linked. Then there exist finite sequences $e = e_0, e_1, \cdots, e_n = f$ and $f_1, f_2, \cdots, f_n$ of primitive idempotents in R such that $f_iRe_{i-1} \neq 0$ and $f_iRe_i \neq 0, 1 \leqq i \leqq n$. By 2.3, Rf_i/Jf_i is simple for each i. So, by 2.2 and the above remark, Rf_i/Jf_i embeds in a factor module of Re_{i-1} and in a factor module of Re_i for each i. Therefore (C) holds.

(2) Assume the hypothesis of (2) and that (C) holds. Then we have a sequence $e = e_0, e_1, \cdots, e_n = f$ of primitive idempotents such that some factor module of Re_{i-1} has a simple in common with a factor module of Re_i for $1 \leqq i \leqq n$. Since R/J is artinian by hypothesis, it is obvious that R has no infinite sets of orthogonal idempotents. As we shall see later in this section (Theorem 2.6), this implies the existence of orthogonal primitive idempotents g_j such that $R = Rg_1 \oplus \cdots \oplus Rg_p$. Hence R/J is isomorphic to $Rg_1/Jg_1 \oplus \cdots \oplus Rg_p/Jg_p$. Consequently, every simple R-module embeds in some Rg_j/Jg_j. But each R-module Rg_j/Jg_j is completely reducible. Thus it follows from 2.2 that for some choice of indices $j_1, j_2, \cdots, j_n$, we have $g_{j_i}Re_{i-1} \neq 0$ and $g_{j_i}Re_i \neq 0$ for $1 \leqq i \leqq n$. Therefore, e and f are linked.

COROLLARY. *If R is a semiperfect ring* (*in the sense of Bass* [1]), *then a necessary and sufficient condition for two primitive idempotents e and f in R to be linked is that condition* (C) *of the proposition holds.*

Proof. If R is semiperfect, then R/J is artinian and primitive idempotents are local ([6], p. 76, Proposition 2).

The next part of this section is devoted to establishing a sufficient condition for the identity of a ring to be a sum of primitive orthogonal idempotents. We omit the proof of the following well known lemma.

LEMMA 2.5. *If R is a ring, then R has no infinite sets of orthogonal idempotents if and only if ${}_RR$ (R_R) has maximal and minimal condition on direct summands.*

THEOREM 2.6. *Any nonzero idempotent in a ring having no infinite sets of orthogonal idempotents is a sum of orthogonal primitive idempotents.*

Proof. Let R be a ring with no infinite sets of orthogonal idempotents and let e be a nonzero idempotent in R. Lemma 2.5 implies the existence of a primitive idempotent $f \in Re$. One checks that ef

and $e - ef$ are orthogonal idempotents. Hence $Re = Ref \oplus R(e - ef)$. But, since $ef \in Rf$ and f is primitive, it is easy to see that ef is primitive. So all that is needed to finish the proof is a standard induction argument using Lemma 2.5.

COROLLARY. *A sufficient condition for the identity of a ring R to be a sum of orthogonal primitive idempotents is for R to contain no infinite sets of orthogonal idempotents.*

REMARK. In a private communication, E. C. Dade has recently provided the author with a counterexample to the converse of the above corollary.

The local idempotent counterpart of the corollary is easily characterized:

LEMMA 2.7. *If R is a ring, then the following statements are equivalent.*

(1) *R is semiperfect.*

(2) *The identity of R is a sum of orthogonal local idempotents.*

(3) *The identity of R is a sum of local idempotents.*

Proof. (1) implies (2). This is [6, p. 76, Corollary 2]. (3) implies (1). The hypothesis of (3) is inherited by R/J in the strong sense that the identity of R/J is a sum of local idempotents of R/J each of which is the canonical image of a local idempotent in R. That is, since R/J is a semiprime ring, it follows that R/J is artinian and that every simple left R-module has the form Re/Je for some local idempotent $e \in R$. For the rest, we merely imitate the proof of a lemma of Bass [1, Lemma 2.6].

Let M be a finitely generated left R-module. The above analysis shows that we may write $M/JM = Re_1/Je_1 \oplus Re_2/Je_2 \oplus \cdots \oplus Re_n/Je_n$ where the e_i are idempotents in R. Since $P = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$ is projective, there exists a map $P \to M$ making the diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr, "nat"] & \\ M \arrow[rr, "nat"'] & & M/JM \end{tikzcd}$$

commutative. One shows easily that $M = JM + \operatorname{im}(P \to M)$. Also, $\ker(P \to M) \subseteqq \ker(P \to M/JM) = JP$. But M and P are both finitely generated. Thus a version of Nakayama's Lemma implies that JM is small in M and JP is small in P. Therefore $P \to M$ is a projective cover. So (1) follows by the definition of semiperfect.

Since (2) $\Rightarrow$ (3) is trivial, we are done.

THEOREM 2.8. *The following two conditions are equivalent in any ring* R.

(1) *Every minimal left ideal in* R *is projective and* R *contains no infinite sets of orthogonal idempotents.*

(2) R *is a ring direct sum of indecomposable rings satisfying condition* (1).

Proof. Suppose R is a ring direct sum, say

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$$

where the R_i are rings. Then every left ideal in R is a direct sum of left ideals in the R_i. Furthermore, R_i as an R_i-module is the "same" as R_i as an R-module. In particular, the left socle of R is the direct sum of the left socles of the R_i. So, by the above argument, the left socle of R is a projective R-module if and only if the left socle of each R_i is a projective R_i-module.

Now each direct summand of R is a direct sum of direct summands of the R_i. It follows that if R fails to satisfy the maximal condition on direct summands, then some R_i must fail to satisfy that condition. Then, by 2.5, R inherits the property of having no infinite sets of orthogonal idempotents from the R_i (note the converse is trivial). So 2.8 follows from 2.1 and 2.6.

3. The case where the socle is projective and essential.

A submodule E of an R-module M is called essential if E intersects every nonzero submodule of M nontrivially. (Note that every essential submodule of M contains the socle of M.) The left singular ideal of a ring R is the set of all elements in R with essential left annihilator. (These definitions are due to R. E. Johnson.)

In the sequel we denote the left annihilator of a subset X of a ring by X^l and the right annihilator of X by X^r.

THEOREM 3.1. *If* R *is a ring with left socle* S, *the following are equivalent.*

(1) S *is a projective, essential submodule of* ${}_RR$.

(2) *The right annihilator of* S *is zero.*

(3) *Some ideal of* R *contained in* S *has zero right annihilator.*

(4) S *is essential in* R *and* R *has zero left singular ideal.*

Proof. (1) $\Rightarrow$ (2). If ${}_RS$ is projective, Lemma 1.1 implies $0 = SS^r = S \cap S^r$. So, if S is also essential, then $S^r = 0$.

$(2) \Rightarrow (3)$. Trivial.

$(3) \Rightarrow (4)$. Let T be an ideal contained in S satisfying $T^r = 0$. If L is a nonzero left ideal, then $0 \neq TL \subseteqq T \cap L$. Therefore, T is essential, implying $T = S$. But then, the left singular ideal of R is clearly just $S^r = 0$.

$(4) \Rightarrow (1)$. Let H be a homogeneous component of S. Obviously $H^2 = SH$. So, if (4) holds, then $H^2 \neq 0$. This implies the projectivity of S.

REMARK. If R happens to have no infinite sets of orthogonal idempotents,[4] then R satisfies (say) condition (1) of Theorem 3.1 if and only if R is a ring direct sum of indecomposable rings each of which satisfies condition (1) of the theorem (see Theorem 2.8).

THEOREM 3.2. *Let R be a ring with projective left socle S. Then there is a family $\{R_\alpha\}_{\alpha \in \Omega}$ of rings with projective, essential and homogeneous left socle with the properties*

(1) *R is a subdirect sum of R/S and the R_α's;*

(2) *if S is essential in R, then R is a subdirect sum of the R_α's;*

(3) *if S has no infinite sets of orthogonal idempotents, then the family $\{R_\alpha\}_{\alpha \in \Omega}$ is finite.*

Proof. Write $S = \bigoplus \sum_{\alpha \in \Omega} S_\alpha$ where the S_α are the homogeneous components of S. Let $P_\alpha = (S_\alpha)^r$ and $R_\alpha = R/P_\alpha$. Since $\bigcap_\alpha P_\alpha = S^r$, Lemma 1.1 implies that $S \cap \bigcap_\alpha P_\alpha = 0$. In particular, S is essential if and only if $\bigcap_\alpha P_\alpha = 0$.

If $x \in R$ is such that $S_\alpha + P_\alpha/P_\alpha \cdot (x + P_\alpha) = 0$, then $S_\alpha(S_\alpha x) = 0$. This implies $S_\alpha x = 0$ (for example by 1.1). That is, $x + P_\alpha = 0$. It follows from Theorem 3.1 that $S_\alpha + P_\alpha/P_\alpha$ is the projective, essential, homogeneous left socle of R_α.

The fact that (3) is immediate by Theorem 1.2 finishes the proof.

THEOREM 3.3. *A ring with projective, essential, finitely generated left socle has maximal and minimal condition on annihilator left and right ideals.*

Proof. Let R be the ring, S its left socle and $\mathscr{Q}$ a nonempty set of annihilator right ideals. Since ${}_RS$ is finitely generated, it is artinian. Hence the set $\{SQ^\ell \mid Q \in \mathscr{Q}\}$ has a minimal element SQ^ℓ, $Q \in \mathscr{Q}$. If Q is not a maximal element of $\mathscr{Q}$, there exists a $P \in \mathscr{Q}$ such that $Q \subset P$. So $Q^\ell \supseteqq P^\ell$ and, consequently, $SQ^\ell \supseteqq SP^\ell$. The

[4] Primitive rings with nonzero socle obviously satisfy (1) in 3.1. However, there are such rings in which the socle has infinite sets of orthogonal idempotents.

minimal property of SQ^{ℓ} implies $SQ^{\ell} = SP^{\ell}$. But, since $S^{r} = 0$ by 3.1, $Q^{\ell r} = P^{\ell r}$. So Q and P, being annihilator right ideals, are equal against $Q \subset P$.

The proof that R has minimal condition on annihilator right ideals is entirely similar.

To finish, we observe that a strictly increasing (decreasing) sequence of annihilator left ideals would lead to a strictly decreasing (increasing) sequence of annihilator right ideals.

QUESTION. What are some natural conditions which force a ring satisfying the hypothesis of Theorem 3.3 to be semiperfect? In § 4 (Example 4.6) we give an example of such a ring which is both left noertherian and artinian modulo its radical; but which is not semiperfect.

THEOREM 3.4. *A left or right perfect ring (in the sense of Bass* [1]) *with finitely generated, projective and essential left socle is semiprimary.*

Proof. We assume first that the ring R is right perfect. By Bass' Theorem P in [1], R/J is artinian. For rings with R/J artinian, it is well known that the socle of the left R-module $R/(J^{i})^{r}$ is $(J^{i+1})^{r}/(J^{i})^{r}$. Again by Theorem P, nonzero left R-modules have nonzero socles. In particular, $(J^{i+1})^{r}/(J^{i})^{r} \neq 0$. We have an ascending sequence $0 \subset J^{r} \subset (J^{2})^{r} \subset \cdots \subset (J^{i})^{r} \subset \cdots$. Therefore, the hypothesis of the theorem implies via 3.3 that $(J^{n})^{r} = R$ for some n i.e., $J^{n} = 0$. So R is semiprimary.

The proof in the left perfect case is analogous.

REMARK. Note that any ring with finitely generated essential left socle trivially has no infinite sets of orthogonal idempotents. Hence the hypothesis in 3.4 that R is left (right) perfect may be weakened to nonzero left (right) R-modules have nonzero socles.

We had originally used Theorem 3.4 to show that a left (or right) perfect ring which embeds in a simple artinian ring must be semiprimary. However, thanks to A. W. Goldie (oral communication), we can give an easy generalization of this result:

THEOREM 3.5. *A necessary condition for a left perfect ring R to embed in a ring with maximal condition on annihilator left ideals is for R to be semiprimary.*

Proof. If R embeds in a ring with maximal condition on annihilator left ideals, a simple argument shows that R must also have maximal condition on annihilator left ideals. But then, the same tactic as used in the proof of Theorem 3.4 shows that R is semiprimary.

4. **Examples.** On the basis of the foregoing material, one might conjecture that the left socle of any ring with nonzero, projective left socle is essential. Indeed, many familiar rings (such as primitive rings with nonzero socle) do have this property. Of course, our "conjecture" is blatantly false. Any ring direct sum of a semisimple artinian ring with a ring having zero left socle is a counterexample. In fact, one may easily characterize such rings.

PROPOSITION 4.1.[5] *Let R be a ring with finitely generated left socle and no nilpotent minimal left ideals. Then R is the ring direct sum of a semisimple artinian ring and a ring with zero left socle.*

Proof. Theorem 1.2 implies the existence of a right ideal H such that $R = S \oplus H$ where S is the left socle. But $HS \subseteqq H \cap S = 0$. Therefore, $(SH)^2 = 0$. Since R has no nilpotent minimal left ideals, $SH = 0$. This implies that H is an ideal.

We would also like to point out that condition (4) of Theorem 3.1 cannot be weakened to read: "R has zero left singular ideal and every minimal left ideal in R is projective." The ring of integers is a counterexample. In fact, any nonartinian semiprime ring with maximal condition on annihilator left ideals has zero left singular ideal and projective, nonessential socle. This follows from 2.5, 4.1 and the fact that a semiprime ring with maximal condition on annihilator left ideals has zero left singular ideal (see the proof in Lambek [6] of Proposition 3, p. 107).

A more sensible conjecture would be that a left perfect ring with nonzero projective left socle has essential left socle. But the ring $P \oplus F$ where P is a left perfect ring with no minimal left ideals and F is any field is a counterexample. Provided, of course, that we can demonstrate the existence of such a perfect ring P. An elegant method of doing this has been kindly donated to us by J. S. Alin: Define inductively $S^0 = 0$, $S^{\alpha+1}$ is such that $S^{\alpha+1}/S^{\alpha}$ is the left socle of R/S^{α}, and, if α is a limit ordinal, $S^{\alpha} = \bigcup_{\beta<\alpha} S^{\beta}$. In this notation, we have

PROPOSITION 4.2. *Every nonzero left R-module has a nonzero socle if and only if $S^{\alpha} = R$ for some ordinal α.*

Proof. "only if". Since R is a set, $S^{\alpha} = S^{\alpha+1}$ for some α. Then

[5] We owe the present form of this proposition to a comment of the referee's.

$0 = S^{\alpha+1}/S^{\alpha} =$ the socle of the left R-module R/S^{α}. So $R = S^{\alpha}$ if nonzero modules have nonzero socles "if". Suppose $S^{\alpha} = R$, some α, and let M be a nonzero left R-module. It is enough to show that $Rx \simeq R/x^{\ell}$ has nonzero socle for $0 \neq x \in M$.

Our hypothesis implies the existence of a smallest β such that $S^{\beta} \subsetneqq x^{\ell}$. Since $S^{\gamma} \subseteqq x^{\ell}$ for $\gamma < \beta$, β cannot be a limit ordinal. Consequently, there is a nonzero homomorphism from $S^{\beta}/S^{\beta-1}$ into R/x^{ℓ}. Since $S^{\beta}/S^{\beta-1}$ is the left socle of $R/S^{\beta-1}$, Rx has nonzero socle.

COROLLARY. *There exists a left perfect local ring with zero left socle.*

Proof. There are familiar examples of left perfect local rings which are not right perfect. Such a ring must have a nonzero left module with zero socle. But a factor ring of a left perfect local ring is a left perfect local ring.

The counterexample above still leaves our "conjecture" open for indecomposable left perfect rings. Suppose R is an indecomposable left perfect ring with nonzero projective left socle. Then R has a nonnilpotent minimal left ideal L. If R is not simple, the corollary to Proposition 2.4 guarantees the existence of at least one principal indecomposable[6] of R which contains a copy of L but is not isomorphic to L. In general, this is the most one can expect.

EXAMPLE 4.3. Let P be a left perfect local ring with zero left socle and set $D = P/J$. Let R be the ring of all matrices of the form

$$\begin{bmatrix} P & 0 & 0 \\ J & P & 0 \\ D & 0 & D \end{bmatrix}.$$

Then R has the following properties.

(1) R is an indecomposable left perfect ring with projective left socle.

(2) $R = \mathscr{P}_1 \oplus \mathscr{P}_2 \oplus \mathscr{P}_3$ where the $\mathscr{P}_i$ are nonisomorphic principal indecomposables. $\mathscr{P}_2$ has zero socle but $\mathscr{P}_1$ and $\mathscr{P}_3$ each have a unique simple submodule (which is isomorphic to $\mathscr{P}_3$).

We leave to the reader the task of showing the example really works.

If an R-module is a finite direct sum of submodules, then it is an essential extension of its socle if and only if the summands are essential extensions of their respective socles (e.g., take injective hulls).

[6] By a principal indecomposable left ideal in a ring R, we mean an indecomposable direct summand of ${}_RR$.

Thus the worst possible example of the failure (at least in left perfect rings) of our "conjecture" would be a left perfect ring with projective nonessential socle and the additional property that every principal indecomposable has nonzero socle. Such an example follows.

EXAMPLE 4.4. An indecomposable left perfect ring R with projective, nonessential socle in which every principal indecomposable has a unique simple submodule:

R is the ring of all matrices of the form $\begin{bmatrix} P & 0 \\ D & D \end{bmatrix}$, the notation being the same as in Example 4.3.

EXAMPLE 4.5. A semiperfect ring R with the properties:

(1) The left socle of R is projective and essential.

(2) Every principal indecomposable of R has a unique simple submodule (in particular, the left socle of R is finitely generated).

(3) R is neither left nor right perfect.

We take a local, commutative integral domain (not a field) L and then take R to be the ring of all matrices of the form $\begin{bmatrix} L & 0 \\ F & F \end{bmatrix}$ where F is the quotient field of L. If R were left or right perfect, Theorem 3.4 would imply that R is semiprimary. This is not so.

EXAMPLE 4.6. A ring R with the properties:

(a) The left socle of R is finitely generated, projective, and essential.

(b) R/J is artinian.

(c) R is not semiperfect.

Let D be a (commutative) noertherian integral domain with only a finite number (greater than one) of maximal ideals. Since D is a Zariski ring with respect to its radical, $D/J(D)$ is artinian [8, p. 264, Th. 10]. Take R to be the ring of all matrices of the form $\begin{bmatrix} D & 0 \\ F & F \end{bmatrix}$ and F is the quotient field of D. R is not semiperfect since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a primitive, nonlocal idempotent of R.

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