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EXTENSION AND BEHAVIOR AT INFINITY OF SOLUTIONS OF CERTAIN LINEAR OPERATIONAL DIFFERENTIAL EQUATIONS

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We consider the linear differential equation $u'' + Bu' + Au = 0$ with coefficients A, B unbounded operators in a Banach space E. Under the assumption that the Cauchy problem for it is well posed in a suitable sense, continuation and behavior at infinity of solutions are studied.

Let E be a complex Banach space, A, B linear operators with domains $D(A)$, $D(B)$ dense in E and range in E. An E-valued function $u(\cdot)$ defined and twice continuously differentiable in $t \geqq 0$ is said to be a *solution* of the operational differential equation

$$u''(t) + Bu'(t) + Au(t) = 0 \tag{1.1}$$

in $[0, \infty[$ if $u(t) \in D(A)$, $u'(t) \in D(B)$, $Au(\cdot)$ and $Bu'(\cdot)$ are continuous functions and (1.1) is satisfied everywhere in $t \geqq 0$. We say that the Cauchy problem for (1.1) is *well posed* in $[0, \infty[$ if

(a) There exist dense subspaces D_0, D_1 of E such that if $u_0 \in D_0$, $u_1 \in D_1$ then there is a solution $u(\cdot)$ of (1.1) with $u(0) = u_0$, $u'(0) = u_1$ (obviously we must have $D_0 \subseteqq D(A)$, $D_1 \subseteqq D(B)$).

(b) For every $t > 0$ there exist constants $K_0(t)$, $K_1(t) < \infty$ such that

$$|u(s)| \leqq K_0(t)\,|u(0)| + K_1(t)\,|u'(0)| \tag{1.2}$$

for $0 \leqq s \leqq t$. Clearly (b) implies uniqueness of solutions of (1.1) with given initial data $u(0)$, $u'(0)$.

We consider in this paper the problem of obtaining *global* estimates for the solutions of (1.1) on the basis of the hypotheses just set forth. We show that, under mild additional restrictions on the solutions of (1.1) there exist constants $K_0, K_1, \omega_0, \omega_1 < \infty$ such that

$$|u(t)| \leqq K_0 e^{\omega_0 t}\,|u'(0)| + K_1 e^{\omega_1 t}\,|u'(0)| \tag{1.3}$$

in $t \geqq 0$, i.e., the solutions of (1.1) increase (at most) exponentially at infinity (Theorem 2.1). This result is analogous to the well known one for first-order equations $u' + Bu = 0$ ([3], Chapter VIII, p. 615) and a generalization of a similar property of the equation $u'' + Au = 0$ ([8], p. 9 and [4], I, p. 90), although the method of proof is different. We next show that, under similar, but slightly stronger, restrictions on the solutions of (1.1) we only need to assume existence and continuous dependence in a *finite* interval $[0, a]$, $a > 0$, that is the solutions can be extended to the positive real axis and satisfy there inequality (1.3) with convenient constants (Theorem 3.1). We then examine, by means of counterexamples the rôle of the additional assumptions on the results of Theorem 2.5 and 3.1. Finally, we sketch the extension of the results to higher-order equations.

It should be noted that if the *derivative* $u'(t)$ of each solution of (1.1) is assumed to depend continuously on its initial data (i.e., if an inequality of the type

$$|u'(t)| \leqq L_0(t)\,|u(0)| + L_1(t)\,|u'(0)|\,, \tag{1.4}$$

$0 \leqq s \leqq t$, $L_0(t)$, $L_1(t) < \infty$ is assumed to hold for any $t > 0$) then the equation (1.1) can be reduced to a first order equation

$$(u_1'(t) = u_2(t),\ u_2'(t) = -Au_1(t) - Bu_2(t))$$

in the product space $E \times E$ to which semigroup theory can be applied and all of the results in this paper can be readily obtained from the corresponding ones for first order equations. However, (1.4) is not satisfied for many of the equations that can be put in the form (1.1) for instance the wave equation. (The author is indebted to the referee for these observations.)

We shall not be concerned here with the problem of finding conditions on the coefficients A, B of (1.1) in order that the Cauchy problem for (1.1) be well posed in some sense or another; for a view on this subject the reader may consult [5], [6], [7] and bibliography therein.

The Cauchy problem for the equation (1.1) has been studied in a similar way but with somewhat different assumptions by M. Sova in [9]; we indicate at several points in this paper the relations between Sova's results and ours.

We hope to present in a forthcoming paper applications of the present results to partial differential equations.

2. We denote by $\mathscr{L}(E)$ the space of all linear bounded operators from E to E, endowed with its customary topology (the "uniform operator" topology). If J is an interval in $]-\infty, \infty[$ and n a nonnegative integer we denote by $C^{(n)}(J, E)$ (or simply $C^{(n)}(J)$) the space of all E-valued functions defined and n times continuously differentiable in J. It is assumed that A, B are such that the Cauchy problem for (1.1) is well posed in $[0, \infty[$; we also suppose that the operators A, B are *closed*.

Let $u \in D_0$. By virtue of (a), § 1, there exists a solution $u(\cdot)$ of (1.1) with initial data

$$u(0) = u \quad u'(0) = 0 \ .$$

Define

$$S(t)u = u(t) \text{ for } t \geqq 0 \ .$$

By virtue of (b), if t is any fixed element of $[0, \infty[$, $S(t)$ is *bounded*. Since D_0 is dense in E, we can extend $S(t)$ to all of E by continuity as a bounded operator (which we shall denote by the same symbol). Because of the estimate (1.2), if $u \in E$, $\{u_n\} \subset D_0$, $u_n \to u$ then $(S(\cdot)u_n \to S(\cdot)u$ uniformly on compacts of $[0, \infty[$. Accordingly $S(\cdot)$ is a strongly continuous application of $[0, \infty[$ into $\mathscr{L}(E)$; $|S(\cdot)|$ is bounded on compacts of $[0, \infty[$ by virtue of (1.2). We define the $\mathscr{L}(E)$-valued function $T(\cdot)$ in the same way, but now in reference to the solution $u(\cdot)$ of (1.1) with initial conditions

$$u(0) = 0, u'(0) = u \in D_1 \ .$$

Clearly $T(\cdot)$ enjoys all of the properties just established for $S(\cdot)$. By definition, we have $S(0) = I$, the identity operator in E, $T(0) = 0$. If $u(\cdot)$ is any solution of (1.1) then

$$u(t) = S(t)u(0) + T(t)u'(0) \ . \tag{2.1}$$

This follows from the very definition of S and T when $u(0) \in D_0$, $u'(0) \in D_1$ and it can be obtained from (1.2) and a passage to the limit in the general case. Observe that if $u \in D_0$, $S(\cdot)u \in C^{(2)}([0, \infty[)$ and $S'(0)u = 0$, $S''(0) = -AS(0)u - BS'(0)u = -Au$; similarly if

$$u \in D_1, \ T(\cdot)u \in C^{(2)}([0, \infty[)$$

and $T'(0) = u$, $T''(0)u = -Bu$. We shall call S, T the *propagators* of (1.1).

The following well-known result will be constantly used in the sequel.

LEMMA 2.0. *Let Q be a closed operator in E with domain $D(Q)$, $f(\cdot)$ a function defined in the (finite or infinite) interval J, with values in $D(Q)$ and such that $f(\cdot)$, $Qf(\cdot)$ are continuous and integrable in J. Then $f = \int_J f(t)dt \in D(Q)$ and*

$$Qf = \int_J Qf(t)dt \ . \tag{2.2}$$

For a proof (of a more general theorem) see for instance [3], Chapter III, p. 153. Our principal result is

THEOREM 2.1. *Assume $T(\cdot)u$ is continuously differentiable in*

$[0, \infty[$ *for all* $u \in E$. *Assume, further, that* $T(t)E \subseteqq D(B)$ *for all* $t \geqq 0$ *and that* $BT(\cdot)u$ *is continuous in* $[0, \infty[$ *for all* $u \in E$. *Then there exist constants* $K_0, K_1, \omega_0, \omega_1 < \infty$ *such that the estimate* (1.3) *holds for every solution* $u(\cdot)$ *of* (1.1).

Proof. It will be carried out by constructing an "approximate resolvent" for the characteristic polynomial $P(\lambda) = \lambda^2 I + \lambda B + A$ of (1.1) by a technique not unlike those of [1], [2] and then by obtaining, by inverse Laplace transform, a convenient functional equation for T.

We examine first a few results that can be immediately drawn from the assumptions in Theorem 2.1 (they will be assumed to hold throughout this section). Let $a > 0$, and assign to the space $C^{(0)}([0, a])$ its usual supremum norm (which makes it a Banach space). The operator $u \rightarrow T'(\cdot)u$ from E to $C^{(0)}([0, a])$ is easily seen to be closed; since it is everywhere defined, by the closed graph theorem it is as well bounded. But then $T'(t)$ is a bounded operator for all t; moreover, the map $t \rightarrow T'(t)$ from $[0, \infty[$ to $\mathscr{L}(E)$ is strongly continuous. By the Banach-Steinhaus theorem $|T'(\cdot)|$ is bounded on compacts of $[0, \infty[$. Consider next the operator $BT(t)$, $t \geqq 0$ from E to E. Again $BT(t)$ is closed and everywhere defined; another application of the closed graph theorem shows that it is bounded. Clearly $BT(\cdot)$ is strongly continuous, $|BT(\cdot)|$ is bounded on compacts of $[0, \infty[$.

We will need later to solve the inhomogeneous equation

$$u''(t) + Bu'(t) + Au(t) = f(t)\ . \tag{2.3}$$

Solutions of (2.3) are defined in the same way solutions of (1.1) are.

LEMMA 2.2. *Let* f *belong to* $C^{(1)}([0, \infty[)$. *Then* (a)

$$\begin{aligned} u(t) &= T(t) * f(t)^{(1)} = \int_0^t T(t-s)f(s)ds \\ &= \int_0^t T(s)f(t-s)ds,\ t \geqq 0 \end{aligned} \tag{2.4}$$

is a solution of (2.3) *in* $t \geqq 0$ *with* $u(0) = u'(0) = 0$.[2] (b) *If* $v(\cdot)$ *is any solution of* (2.3) *then*

$$v(t) = S(t)v(0) + T(t)v'(0) + u(t),\ t \geqq 0 \tag{2.5}$$

where u *is defined by* (2.4).

Proof. Integrating (2.4) by parts we obtain

(1) We shall denote occasionally a function f, or $f(\cdot)$ in the same way we denote one of its values ($f(t)$). This will cause no confusion.

(2) See [**9**], p. 99 for a related result.

$$u(t) = \int_0^t T(s)f(0)ds + \int_0^t\left(\int_0^{t-s} T(r)dr\right)f'(s)ds .$$

Differentiating,

$$u'(t) = T(t)f(0) + \int_0^t T(t-s)f'(s)ds ,$$

$$u''(t) = T'(t)f(0) + \int_0^t T'(t-s)f'(s)ds$$

(the foregoing steps can be easily justified). Let now $u \in D_1$. We have

$$AT(s)u = -BT'(s)u - T''(s)u$$

or, integrating between 0 and t,

$$A\int_0^t T(s)u\,ds = -BT(t)u - T'(t)u + u .$$

Since the right-hand side of the preceding equality depends continuously on u and A is closed, it follows from denseness of D_1 that

$$\int_0^t T(s)u\,ds \in D(A)$$

for all $u \in E$ and that

$$U(t) = A\int_0^t T(s)ds + BT(t) + T'(t) \equiv I . \tag{2.6}$$

The preceding observations and Lemma 2.0 make clear that $u(t) \in D(A)$ and that $Au(\cdot)$ is a continuous function. Similarly, the assumption on $BT(\cdot)$ in Theorem 2.1 implies that $u'(t) \in D(B)$ and that $Bu'(\cdot)$ is a continuous function. Finally, it is plain that

$$\begin{aligned} u''(t) + Bu'(t) + Au(t) &= U(t)f(0) \\ &+ \int_0^t U(t-s)f'(s)ds = f(t) \end{aligned}$$

as claimed.

Observe, finally, that if $v(\cdot)$ is an arbitrary solution of (2.3), $u(\cdot)$ the solution provided by formula (2.4) then $v(\cdot) - u(\cdot)$ is a solution of (1.1). Making use of (2.1) we obtain the formula (2.5).

LEMMA 2.3. (a) *Let* $u \in D(A)$. *Then*

$$S'(t)u = -T(t)Au \tag{2.7}$$

(b) *Let* $u \in D_0 \cap D(B)$. *Then*

$$T'(t)u = S(t)u - T(t)Bu \,. \tag{2.8}$$

Proof. (a) According to Lemma 2.2 $u(t) = -\int_0^t T(s)Au\,ds$ is a solution of the equation

$$u''(t) + Bu'(t) + Au(t) = -Au$$

with $u(0) = u'(0) = 0$. Consequently $v(t) = u(t) + u$ satisfies (1.1) with initial conditions $v(0) = u$, $v'(0) = 0$. By virtue of (2.1), $v(t) = S(t)u$, that is

$$S(t)u - u = -\int_0^t T(s)Au\,ds \tag{2.9}$$

which is the integrated version of (2.7). As for (b), let now

$$u(t) = -\int_0^t T(s)Bu\,ds \,.$$

Applying again Lemma 2.2 we see that $u(\cdot)$ satisfies

$$u''(t) + Bu'(t) + Au(t) = -Bu \tag{2.10}$$

$u(0) = u'(0) = 0$. On the other hand, let $v(t) = \int_0^t S(s)u\,ds$. We have $v'(t) = S(t)u = \int_0^t S'(s)u\,ds + u$, $v''(t) = S'(t)u = \int_0^t S''(s)u\,ds$. Making use of Lemma 2.0 we easily see that $v(\cdot)$ satisfies as well (2.10)-but now with initial conditions $v(0) = 0$, $v'(0) = u$. Accordingly $w(t) = v(t) - u(t)$ satisfies (1.1) with initial conditions $w(0) = 0$, $w'(0) = u$, that is, $w(t) = T(t)u$, or

$$T(t)u = \int_0^t (S(s)u - T(s)Bu)ds \tag{2.11}$$

from which (2.8) can be deduced by differentiation.

COROLLARY 2.4. (a) $D_0 = D(A)$. (b) $D = D(A) \cap D(B)$ *is dense in* E. (c) $D_1 \supseteqq D(A) \cap D(B)$[3]. (d) *If* $u \in D(A) \cap D(B)$,

$$T''(t)u + T'(t)Bu + T(t)Au = 0,\ t \geqq 0 \,.$$

Proof. As a by-product of the proof of Lemma 2.3 (a) it was established that $S(\cdot)u$ is a solution of (1.1) for any $u \in D(A)$. Similarly, one of the steps in the proof of (b) was to show that

$$T(\cdot)u,\ u \in D_0 \cap D(B) = D(A) \cap D(B)$$

is a solution of (1.1). To show (b), let Ψ be the subspace of E generated by all elements of the form

[3] It is *not* true in general that $D_1 = D(B)$.

$$\int \Psi(s) T'(s) u\, ds$$

where Ψ is, say, any C^∞ function with compact support contained in $]0, \infty[$, u any element of E. Making use of the fact that $T'(0) = I$ it is simple to show that Ψ is dense in E (see [3], Chapter VIII, Exercise 3.1 for a similar statement). On the other hand, we observe that (integrating by parts) any element of Ψ can be written in the form

$$-\int \Psi'(s) T(s) u ds = \int \Psi''(s) \left(\int_0^s T(r) u dr \right) ds .$$

Applying Lemma 2.0 to the first of these expressions we obtain $\Psi \subseteqq D(B)$; on the other hand, again by Lemma 2.0, equality (2.6) and the comments preceding it, $\Psi \subseteqq D(A)$, which establishes (b). As for (d), it immediately follows from differentiating (2.8) and then expressing $S'(t)u$ by means of (2.7).

We may remark at this point that, as a consequence of equality (2.8) the operator $T(t)B$ (with $D(A) \cap D(B)$ as domain) admits a bounded extension to all of E, namely

$$\overline{T(t)B} = S(t) - T'(t) . \tag{2.12}$$

Since $S(\cdot)$, $T'(\cdot)$ are strongly continuous functions in $]0, \infty[$, so is $\overline{T(\cdot)B}$.

We consider in what follows the "characteristic polynomial"

$$P(\lambda) = \lambda^2 I + \lambda B + A$$

of (1.1); for each λ, $P(\lambda)$ is a linear operator with domain

$$D = D(A) \cap D(B) .$$

LEMMA 2.5. (a) *$P(\lambda)$ is pre-closed for all λ.* (b) *There exist constants $\alpha, \beta \geqq 0$ such that $P(\lambda)$ is one-to-one for*

$$\operatorname{Re} \lambda \geqq \alpha + \beta \log(1 + |\lambda|) .\text{[4]} \tag{2.13}$$

Proof. Assume (a) is false for some λ. Then there exists a sequence $\{u_n\} \subset D(P(\lambda))$ such that $u_n \to 0$, $v_n = P(\lambda)u_n \to v \neq 0$. Let $u_n(t) = e^{\lambda t}u_n$, $t \geqq 0$. Clearly $u_n(\cdot)$ satisfies the inhomogeneous equation (2.3) with $f(t) = e^{\lambda t}v_n$. We get as a consequence of Lemma 2.2 (b) that

[4] $P(\lambda)$ may not be closed or one-to-one for some values of λ.

$$e^{\lambda t}u_n = S(t)u_n + \lambda T(t)u_n + \int_0^t T(s)e^{\lambda(t-s)}v_n ds .$$

Letting now $n \to \infty$, we obtain

$$\int_0^t e^{-\lambda s}T(s)vds = 0,\ t \geqq 0 .$$

Differentiating twice

$$e^{-\lambda s}T'(s)v - \lambda e^{-\lambda s}T(s)v = 0 \text{ for } s \geqq 0 ;$$

if we set $s = 0$ in this last expression we obtain $v = 0$, absurd.

As for (b), assume $P(\lambda)$ is not one-to-one for some λ. Then there exists $u \in D(P(\lambda))$, $u \neq 0$ such that $P(\lambda)u = 0$. Obviously $u(t) = e^{\lambda t}u$ is a solution of (1.1); making use of the estimate (1.2) for any fixed $t > 0$ we see that there exists a constant $K < \infty$ such that

$$e^{(\mathrm{Re}\lambda)t} \leqq K(1 + |\lambda|) .$$

Taking logarithms we obtain the inequality opposite to (2.13) for $\alpha = (\log K)/t$, $\beta = 1/t$.

We continue now the proof of Theorem 2.1. Let φ be a twice continuously differentiable scalar valued function with compact support and such that $\varphi(0) = 1$. Consider the (plainly bounded) operator in E

$$R(\lambda, \varphi)u = \int_0^\infty e^{-\lambda t}\varphi(t)T(t)udt \tag{2.14}$$

defined for all complex λ. We easily obtain from Lemma 2.0 that $R(\lambda, \varphi)E \subseteqq D(B)$. Moreover, we can write integrating by parts

$$R(\lambda, \varphi)u = \int_0^\infty (e^{-\lambda t}\varphi(t))'\left(\int_0^t T(s)u\,ds\right)dt \tag{2.15}$$

and then it follows (again from Lemma 2.0, equality (2.6) and the comments preceding it) that $R(\lambda, \varphi)E \subseteqq D(A)$. Hence

$$R(\lambda, \varphi)E \subseteqq D = D(A) \cap D(B) = D(P(\lambda)) . \tag{2.16}$$

If $u \in D$ we easily obtain, after a few integrations by parts and using the fact that $T(\cdot)u$ is a solution of (1.1) (Corollary 2.4, (b))

$$\begin{aligned} P(\lambda)R(\lambda, \varphi)u &= u + \int_0^\infty e^{-\lambda t}[(\varphi T)''(t)u + B(\varphi T)'u + A(\varphi T)u]dt \\ &= u + \int_0^\infty e^{-\lambda t}M(t, \varphi)udt = u + \hat{M}(\lambda, \varphi)u \end{aligned} \tag{2.17}$$

where $M(t, \varphi) = 2\varphi'(t)T'(t) + \varphi''(t)T(t) + \varphi'(t)BT(t)$ is plainly a $\mathscr{L}(E)$-

valued, strongly continuous function in $[0, \infty[$ with compact support. Let $\omega \geqq 0$ be such that

$$\int_0^\infty e^{-\omega t}|M(t, \varphi)|\,dt = \gamma < 1 . \tag{2.18}$$

Plainly

$$|\hat{M}(\lambda, \varphi)| \leqq \gamma \tag{2.19}$$

in $\operatorname{Re}\lambda \geqq \omega$ and thus $I + \hat{M}(\lambda, \varphi)$ has a bounded inverse there. Define

$$\begin{aligned} R(\lambda) &= R(\lambda, \varphi)(I + \hat{M}(\lambda, \varphi))^{-1} \\ &= R(\lambda, \varphi) \sum_{n=0}^{\infty} (-1)^n \hat{M}(\lambda, \varphi)^n . \end{aligned} \tag{2.20}$$

We now write (2.17) in the form

$$\overline{P(\lambda)}R(\lambda, \varphi)u = u + \hat{M}(\lambda, \varphi)u,\ u \in D \tag{2.21}$$

where $\overline{P(\lambda)}$ denotes the closure of $P(\lambda)$. It follows immediately from the fact that $\overline{P(\lambda)}$ is closed that (2.21) must hold as well for all $u \in E$. Then

$$\overline{P(\lambda)}R(\lambda)u = u,\ u \in E . \tag{2.22}$$

Observe now that, since $R(\lambda, \varphi)E \subseteqq D(P(\lambda))$, $R(\lambda)E \subseteqq D(P(\lambda))$; hence, we may replace $\overline{P(\lambda)}$ by $P(\lambda)$ in (2.22). The equality thus obtained implies that $R(\lambda)E = D(P(\lambda))$, at least for those values of λ for which $P(\lambda)$ is one-to-one. For, let $v \in D(P(\lambda))$, $v \notin R(\lambda)E$ and let $u = P(\lambda)v$. Then

$$P(\lambda)(v - R(\lambda)u) = 0$$

which is impossible. We show next that $P(\lambda)$ is actually one-to-one in $\operatorname{Re}\lambda > \omega$. Observe first that $R(\cdot, \varphi)$, $\hat{M}(\cdot, \varphi)$ are *entire* functions, as Laplace transforms of functions with compact support. But then, by virtue of the estimate (2.19) the series in the right-hand side of (2.20) converges uniformly in $\operatorname{Re}\lambda \geqq \omega$, hence $R(\lambda)$ is analytic there. Let now $v \in D(P(\lambda)) = D$, $v \neq 0$ and let λ be, say, in the region defined by (2.13). By the preceding comments, $v = R(\lambda)u(\lambda)(u(\lambda)$ some element in E). Then

$$R(\lambda)P(\lambda)v = R(\lambda)P(\lambda)R(\lambda)u(\lambda) = R(\lambda)u(\lambda) = v . \tag{2.23}$$

The left-hand side of (2.23) is analytic in $\operatorname{Re}\lambda > \omega$. Since it equals v in the region defined by (2.13) it must equal v as well in $\operatorname{Re}\lambda > \omega$, which shows that $P(\lambda)v \neq 0$ throughout $\operatorname{Re}\lambda > \omega$ as claimed. Collecting all the observations made about $P(\lambda)$ and $R(\lambda)$ we can write

$$R(\lambda) = P(\lambda)^{-1} \text{ in } \operatorname{Re}\lambda > \omega . \tag{2.24}$$

We obtain now some rough estimates for R, BR, AR in $\operatorname{Re}\lambda \geqq \omega$. Plainly $|R(\lambda, \varphi)|$, $|BR(\lambda, \varphi)|$ are bounded there; on the other hand, it follows from (2.15) that $|AR(\lambda; \varphi)| \leqq C|\lambda|$. Finally, in view of (2.19)

$$|(I + \hat{M}(\lambda, \varphi))^{-1}| \leqq \sum_{n=0}^{\infty} \gamma^n = (1-\gamma)^{-1} . \tag{2.19}$$

Accordingly,

$$|R(\lambda)|, |BR(\lambda)| \leqq C, \; |AR(\lambda)| \leqq C|\lambda| \tag{2.25}$$

in $\operatorname{Re}\lambda > \omega$ for some convenient constant C. Let now $\bar{\omega} > \omega$ and let $u \in E$. Define

$$u(t) = \frac{1}{2\pi i}\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty} \frac{e^{\lambda t}}{\lambda^4} R(\lambda) u \, d\lambda . \tag{2.26}$$

It is clear that $u(\cdot) \in C^{(2)}([0, \infty[)$, for differentiation under the integral sign is permissible. More precisely, we have

$$u^{(k)}(t) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\bar{\omega}+i\infty} \frac{e^{\lambda t}}{\lambda^{4-k}} R(\lambda) u d\lambda \tag{2.27}$$

$k = 1, 2$. Using now the estimates (2.25) together with Lemma 2.0 we see that $u(t) \in D(A)$, $u'(t) \in D(B)$, $Au(t)$ and $Bu'(t)$ are continuous functions in $t \geqq 0$; we easily compute $u(0) = u'(0) = 0$. In addition, we have,

$$\begin{aligned} u''(t) + Bu'(t) + Au(t) &= \frac{1}{2\pi i}\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty} \frac{e^{\lambda t}}{\lambda^4} P(\lambda) R(\lambda) u d\lambda \\ &= \left(\frac{1}{2\pi i}\int_{\bar{\omega}+i\infty}^{\bar{\omega}+i\infty} \frac{e^{\lambda t}}{\lambda^4} d\lambda\right) u \\ &= \left(\frac{1}{3!}\left(\frac{d}{d\lambda}\right)^3 e^{\lambda t}\right) u = \frac{t^3}{3!} u . \end{aligned} \tag{2.28}$$

Expressing now the solution of (2.28) by means of Lemma 2.2 we obtain

$$\frac{1}{3!}\int_0^t (t-s)^3 T(s) u \, ds = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{e^{\lambda t}}{\lambda^4} R(\lambda) u d\lambda \tag{2.29}$$

a formula that suggests-as will be proved later-that $R(\lambda)$ is the Laplace transform of T.

We now try to find a new representation for $R(\lambda)$. Let $u \in D$; operating as in (2.17) and making use of Corollary 2.4 (d) we can write

$$\begin{aligned} R(\lambda; \varphi) P(\lambda) u = u + \int_0^\infty e^{-\lambda t} [(\varphi T)''(t) u + (\varphi T)'(t) B u \\ + (\varphi T)(t) A u] dt \end{aligned} \tag{2.30}$$

$$= u + \int_0^\infty e^{-\lambda t} N(t; \varphi) u \, dt = u + \hat{N}(\lambda; \varphi) u$$

where now $N(t, \varphi) = 2\varphi'(t)T'(t) + \varphi''(t)T(t) + \varphi'(t)\overline{T(t)B}$. If $\omega' \geqq 0$ is such that

$$\int_0^\infty e^{-\omega' t} |N(t, \varphi)| \, dt = \gamma < 1 \tag{2.31}$$

then $|\hat{N}(\lambda, \varphi)| \leqq \gamma$ in $\operatorname{Re}\lambda \geqq \omega'$, $I + \hat{N}(\lambda, \varphi)$ is invertible there. Let $Q(\lambda) = (I + \hat{N}(\lambda, \varphi))^{-1} R(\lambda, \varphi)$. It follows from (2.30) that

$$Q(\lambda)P(\lambda)u = u \tag{2.32}$$

for $u \in D$, which plainly shows that $Q(\lambda) = R(\lambda)$ in

$$\operatorname{Re}\lambda \geqq \omega_1 = \max(\omega, \omega') \text{ .[5]}$$

Accordingly,

$$\begin{aligned} R(\lambda) &= (I + \hat{N}(\lambda, \varphi))^{-1} R(\lambda, \varphi) \\ &= \Big(\sum_{n=0}^{\infty} (-1)^n \hat{N}(\lambda, \varphi)^n \Big) R(\lambda, \varphi) \, . \end{aligned} \tag{2.33}$$

Formula (2.33) suggests, by inversion of Laplace transforms (as yet formally!) the equality

$$T(t) = \Big(\sum_{n=0}^{\infty} (-1)^n N(t, \varphi)^{*n} \Big) * (\varphi T)(t) \tag{2.34}$$

where $*$ denotes convolution, the exponent $*n$ indicates the n-th convolution power. We attempt to justify now (2.34) directly. By virtue of (2.31) and of Young's inequality

$$\int_0^\infty e^{-\omega' t} |N(t, \varphi)^{*2}| dt \leqq \int_0^\infty |e^{-\omega' t} N(t, \varphi)|^{*2} dt \leqq \gamma^2, \cdots$$

and in general

$$\int_0^\infty e^{-\omega' t} |N(t, \varphi)^{*n}| dt \leqq \gamma^n, \ n \geqq 1 \, .$$

If now K is a constant such that

$$|N(t, \varphi)| \leqq K e^{\omega' t}, \ t \geqq 0 \, ,$$

it is clear that

$$|N(t, \varphi)^{*n}| = |N(t, \varphi)^{*(n-1)} * N(t, \varphi)| \leqq K \gamma^{n-1} e^{\omega' t}, \ n \geqq 1 \, . \tag{2.35}$$

[5] We might set here $\omega_1 = \min(\omega, \omega')$; for if $\omega' < \omega$, it is not difficult to see that $R(\lambda)$-that can be analytically continued to $\operatorname{Re}\lambda > \omega'$ by means of $Q(\lambda)$-still satisfies $R(\lambda) = P(\lambda)^{-1}$ there.

Consequently, the series

$$\sum_{n=1}^{\infty}(-1)^n N(t,\varphi)^{*n} \tag{2.36}$$

converges uniformly on compacts of $[0, \infty[$ to a $\mathscr{L}(E)$-valued function $\mathscr{N}(t, \varphi)$ such that

$$|\mathscr{N}(t,\varphi)| \leqq K(1-\gamma)^{-1}e^{\omega' t},\ t \geqq 0 \tag{2.37}$$

moreover, since each of the terms of the series is strongly continuous in $[0, \infty[$, so is $\mathscr{N}(\cdot, \varphi)$. By virtue of (2.37) and (2.35) the Laplace transform of $\mathscr{N}(\cdot, \varphi)$ exists for $\operatorname{Re}\lambda > \omega'$ and can be computed by term-by-term integration of (2.36). Let now $\tilde{T}$ be the $\mathscr{L}(E)$-valued function defined by the right-hand side of (2.34), that is

$$\begin{aligned}\tilde{T}(t) &= (\delta \otimes I + \mathscr{N}(t,\varphi))*(\varphi T)(t)\\ &= (\varphi T)(t) + \int_0^t \mathscr{N}(t-s,\varphi)(\varphi T)(s)ds\,.\end{aligned} \tag{2.38}$$

Plainly

$$|\tilde{T}(t)| \leqq K' e^{\omega' t},\ t \geqq 0 \tag{2.39}$$

for some constant K'. Computing the Laplace transform of $\tilde{T}$ by application of the convolution theorem, and likewise applying the convolution theorem to each of the terms in the series of $\mathscr{N}(\cdot, \varphi)$ we easily see that it equals

$$\left(\sum_{n=0}^{\infty}(-1)^n \hat{N}(\lambda,\varphi)^n\right)R(\lambda,\varphi) = R(\lambda)$$

by (2.33). But then, by a well-known result on Laplace transforms of antiderivatives, we have

$$\frac{1}{3!}\int_0^t (t-s)^3 \tilde{T}(s)u\,ds = \frac{1}{2\pi i}\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty}\frac{e^{\lambda t}}{\lambda^4}R(\lambda)u\,d\lambda$$

for $\bar{\omega} > \omega_1$, $u \in E$. Comparing this with (2.29) and differentiating three times the identity obtained therefrom by uniqueness of Laplace transforms we obtain $\tilde{T} = T$. In view of (2.39),

$$|T(t)| \leqq K' e^{\omega' t},\ t \geqq 0 \tag{2.40}$$

as we desired to show. Apply now both sides of (2.38) to an arbitrary element of E and differentiate; taking into account that $(\varphi T)(0) = 0$, we obtain

$$T'(t) = (\delta \otimes I + \mathscr{N}(t,\varphi))*(\varphi T)'(t)\,.^{(6)} \tag{2.41}$$

(6) We are using here the differentiation formula

$$(f(t)*g(t))' = f'(t)*g(t) + f(0)g(t) = f(t)*g'(t) + f(t)g(0)\,,$$

valid when f on g are (say) continuously differentiable in $t \geq 0$, zero in $t < 0$.

Similarly, applying both sides of (2.38) to all elements of E of the form $u = Bv$ and then making use of equality (2.12) and the comments preceding it, we get

$$S(t) - T'(t) = (\delta \otimes I + \mathscr{N}(t, \varphi)) * ((\varphi S)(t) - (\varphi T')(t)) . \tag{2.42}$$

Adding now the preceding inequality to (2.41), the equality

$$S(t) = (\delta \otimes I + \mathscr{N}(t, \varphi)) * ((\varphi S)(t) + (\varphi' T)(t)) \tag{2.43}$$

follows. Equalities (2.41), (2.43) can now be used in conjunction with (2.37) to prove that

$$|S(t)| \leqq K'' e^{\omega' t}, \; |T'(t)| \leqq K'' e^{\omega' t} \tag{2.44}$$

in $t \geqq 0$, which ends the proof of Theorem 2.1. (Note that as a by-product of the proof we have established exponential increase of $T'(\cdot)$).

REMARK 2.6. There are "left-handed" analogues of identities (2.38) and (2.41). They are

$$T(t) = (\varphi T)(t) * (\delta \otimes I + \mathscr{M}(t, \varphi)) \tag{2.45}$$

$$T'(t) = (\varphi T)'(t) * (\delta \otimes I + \mathscr{M}(t, \varphi)) \tag{2.46}$$

where

$$\mathscr{M}(t, \varphi) = \sum_{n=1}^{\infty} (-1)^n M(t, \varphi)^{*n} . \tag{2.47}$$

These formulas can be justified along the lines the "right-handed" formulas were. On the basis of (2.18) it can be shown that

$$|M(t, \varphi)^{*n}| \leqq L\gamma^{n-1} e^{\omega t}, \; n \geqq 1$$

for some L and thus that the series in (2.47) converges uniformly on compacts of $[0, \infty[$, its limit $\mathscr{M}$ satisfying

$$|\mathscr{M}(t, \varphi)| \leqq L(1 - \gamma)^{-1} e^{\omega t}, \; t \geqq 0$$

for some constant L. The equality of the left and right-hand side of (2.45) can be eatablished, as in the case of (2.38) by taking the Laplace transform of both sides and then using (2.20) and (2.29). Formula (2.46) can be deduced by applying both sides of (2.45) to an arbitrary element of E and then differentiating. Formulas (2.45) and (2.46) can be used to show that

$$|T(t)| \leqq L' e^{\omega t}, \quad |T'(t)| \leqq L' e^{\omega t} \tag{2.48}$$

for some constant L', $t \geqq 0$, which may or may not be an (asymptotic) improvement on inequalities (2.40), (2.44) for T, T' according to whether or not $\omega > \omega'$. We can, however, obtain a new result from (2.45); pre-multiplying it by B, we get

$$BT(t) = (\varphi BT)(t) * (\delta \otimes I + \mathscr{M}(t, \varphi)) . \tag{2.49}$$

As a consequence,

$$|BT(t)| \leqq L'' e^{\omega t} \tag{2.50}$$

in $t \geqq 0$ and a convenient constant L'', an unscheduled result.

REMARK 2.7. As a by-product of the proof we have obtained some information about the characteristic polynomial $P(\lambda)$ of (1.1); $P(\lambda)$ is closable for all λ, closed in a half-plane $\operatorname{Re} \lambda > \omega_1$ and with a bounded inverse $R(\lambda)$ there that depends analytically on λ, etc.

REMARK 2.8. Among all of M. Sova's results in [9] about the equation (1.1) there is one that is closely related with ours. Roughly speaking, Sova gives a necessary and sufficient condition on $R(\lambda)$ (of the "Hille-Yosida-Phillips" type) for the Cauchy problem for (1.1) to be well set in $[0, \infty[$ *and* for an estimate of the type of (1.3) to hold. (See [9], especially Theorems 6.1 and 6.2.) It might be remarked that, although exponential increase of the solutions is assumed at the outset, no condition of the type of ours (boundedness of T', BT, etc.) is assumed.

REMARK 2.9. Using time independence of the coefficients of (1.1) a number of identities concerning its propagators can be easily derived. Although they will not be used in what follows (except in § 4) we give two examples. Let $t \geqq 0$ fixed, $u \in D_0 = D(A)$ and consider

$$u(s) = S(s + t)u, \ s \geqq 0 .$$

Since $u(\cdot)$ is a solution of (1.1) we obtain, applying (2.1) and using (2.7) to compute $u'(0)$, that

$$S(s + t)u = S(s)S(t)u - T(s)T(t)Au . \tag{2.51}$$

This shows, in particular, that $T(s)T(t)A$ admits a bounded extension to all of E (namely, $S(s)S(t) - S(s + t)$). Reasoning in the same way with $u(s) = T(s + t)u$, $s \geqq 0$, $u \in D(A) \cap D(B)$, we obtain

$$T(s + t)u = S(s)T(t)u + T(s)S(t)u - T(s)T(t)Bu . \tag{2.52}$$

3. We examine here the case in which the Cauchy problem for

$$u''(t) + Bu'(t) + Au(t) = 0 \tag{3.1}$$

is well posed-but only in a *finite* interval $[0, a]$, $a > 0$. Solutions of (3.1) exist for initial data u_0, u_1 in dense subspaces D_0, D_1-although a priori only for t in $[0, a]$-and an estimate of the form

$$|u(t)| \leqq K_0 |u(0)| + K_1 |u'(0)|, \ 0 \leqq t \leqq a$$

is assumed to hold for all solutions. The operators $S(\cdot)$, $T(\cdot)$ of § 2 are now only defined in $[0, a]$, but all the results concerning them in § 2 are valid in this restricted range of t. The proofs are identical.

Throughout this section we write $D = D(A) \cap D(B)$; but now

$$D_2 = \{u \in D; Bu \in D\} .$$

THEOREM 3.1. *Let the Cauchy problem for* 3.1 *be well posed in* $[0, a]$, $a > 0$, *and let* D_2 *be dense in* E. *Assume that* $T(\cdot)u$ *is continuously differentiable in* $[0, a]$ *for all* $u \in E$. *Assume, further, that* $T(t)E \subseteqq D(B)$ *and that* $BT(t)u$ *is continuous in* $[0, a]$ *for all* $u \in E$. *Then the Cauchy problem for* (3.1) *is well posed in* $[0, \infty[$ *and there exist constants* $K_0, K_1, \omega_0, \omega_1 < \infty$ *such that*

$$|u(t)| \leqq K_0 e^{\omega_0 t} |u(0)| + K_1 e^{\omega_1 t} |u'(0)| \tag{3.2}$$

for all solutions $u(\cdot)$ *of* (3.1).

Proof. It will be carried out by slightly modifying that of Theorem 2.1. (It should be pointed out that, due to the additional hypothesis of denseness of D_2 Theorem 3.1 does *not* generalize Theorem 2.1.) Observe first that the operator $R(\lambda) = P(\lambda)^{-1}$, $\operatorname{Re}\lambda > \omega$ was constructed there making use of the values and properties of T *only in the support of* φ; all the auxiliary results, like Lemmas 2.2, 2.3 and 2.5, Corollary 2.4, can be proved in these conditions. Hence the first part of the proof of Theorem 3.1 can be mimicked here if only we take $\operatorname{supp}(\varphi) \subseteqq [0, a]$. The main difference consists in that we will now use (some of) the identities (2.38), (2.41), (2.43) and their "left-handed" analogues (2.45), (2.46) not to *represent* S, T in $[0, \infty[$-they are not a priori defined there-but to *extend* them. Because of this, a somewhat more careful (and tedious) handling of these identities becomes necessary. We shall assume in what follows that the auxiliary function φ used in the construction of $R(\lambda)$ is actually *four* times continuously differentiable; in addition of the condition $\varphi(0) = 1$, we shall also suppose that $\varphi'(0) = \varphi''(0) = 0$. This will simplify some computations later on. Let

$$\tilde{T}(t) = (\varphi T)(t) + \mathscr{N}(t, \varphi) * (\varphi T)(t) \tag{3.3}$$

where $\mathscr{N}$, as in § 2, is defined by the series (2.36). Just as in that section, it can be proved that $\tilde{T}(\cdot)$ is a strongly continuous $\mathscr{L}(E)$-

valued function satisfying (2.39). $\tilde{T}(\cdot)$ can also be defined as

$$\tilde{T}(t) = (\varphi T)(t) + (\varphi T) * \mathscr{M}(t, \varphi) \tag{3.4}$$

the identity between the functions defined by (3.3) and (3.4) being a consequence of the fact that both have $R(\lambda)$ as Laplace transform.

Let now $u \in D_2$. By virtue of Corollary (2.4)

$$T''(t)u = -T(t)Au - T'(t)Bu . \tag{3.5}$$

Since $Bu \in D \subseteqq D_1$ we can differentiate (3.5) once more, then $T(\cdot)u \in C^{(3)}([0, a])$. On the other hand, $BT'(\cdot)Bu$ is a continuous function-again we are using the fact that $Bu \in D_1$; after (3.5) so is $BT''(\cdot)u$. An application of Lemma 2.0 yields

$$BT(t)u = \int_0^t (t - s)BT''(s)u\,ds + tBu$$

which plainly shows that $BT(\cdot)u \in C^{(2)}([0, a])$. Accordingly,

$$M(t, \varphi)u = 2(\varphi' T')(t)u + (\varphi'' T)(t)u + (\varphi' BT)(t)u$$

belongs to $C^{(2)}([0, a])$ if $u \in D_2$. Evidently the same is true of

$$N(t, \varphi)u = 2(\varphi' T')(t)u + (\varphi'' T)(t)u + (\varphi' T)(t)Bu .$$

As a last preliminary step, we modify slightly (3.3) and (3.4). Observe that we can write

$$\mathscr{N}(t, \varphi) = -N(t, \varphi) - \mathscr{N}(t, \varphi) * N(t, \varphi) \tag{3.6}$$

$$\mathscr{M}(t, \varphi) = -M(t, \varphi) - \mathscr{M}(t, \varphi) * M(t, \varphi) \tag{3.7}$$

the justification of (3.6) and (3.7) residing in the fact that the series (2.36) defining $\mathscr{N}$ and the series (2.47) defining $\mathscr{M}$ converge uniformly on compacts of $[0, \infty[$ and can thus be "convoluted term by term" by N and M respectively. Replacing (3.6) in (3.3) we obtain

$$\begin{aligned} \tilde{T}(t) = (\varphi T)(t) &- N(t, \varphi) * (\varphi T)(t) \\ &- \mathscr{N}(t, \varphi) * N(t, \varphi) * (\varphi T)(t) . \end{aligned} \tag{3.8}$$

We apply now both sides of (3.8) to an element $u \in D_2$. Making use of the preceding remarks we obtain[7]

$$\begin{aligned} \tilde{T}'(t)u = (\varphi T)'(t)u &- N(t, \varphi) * (\varphi T)'(t)u \\ &- \mathscr{N}(t, \varphi) * N(t, \varphi) * (\varphi T)'(t)u \end{aligned}$$

$$\begin{aligned} \tilde{T}''(t)u = (\varphi T)''(t)u &- N(t, \varphi)u - N(t, \varphi) * (\varphi T)''(t)u \\ &- \mathscr{N}(t, \varphi) * N(t, \varphi)u \\ &- \mathscr{N}(t, \varphi) * N(t, \varphi) * (\varphi T)''(t)u \end{aligned}$$

[7] See Footnote (6), § 2.

$$\begin{aligned}\tilde{T}'''(t)u = (\varphi T)'''(t)u &- N'(t, \varphi)u \\ &+ N(t, \varphi)Bu - N(t, \varphi) * (\varphi T)'''(t)u \\ &- \mathscr{N}(t, \varphi) * N'(t, \varphi)u + \mathscr{N}(t, \varphi) * N(t, \varphi)Bu \\ &- \mathscr{N}(t, \varphi) * N(t, \varphi) * (\varphi T)'''(t)u,\ t \geqq 0\end{aligned}$$

(we have used at various points the identities $N(0, \varphi)u = (\varphi T)(0)u = 0$, $(\varphi T)'(0)u = u$, $(\varphi T)''(0)u = -Bu$, the last two being a consequence of the fact that $\varphi'(0) = 0$). Consequently $\tilde{T}(\cdot)u \in C^{(3)}([0, \infty[)$ for $u \in D_2$. We turn now to (3.4); replacing (3.7) in it we obtain

$$\tag{3.9} \begin{aligned}\tilde{T}(t) = (\varphi T)(t) &- (\varphi T)(t) * M(t, \varphi) \\ &- (\varphi T)(t) * \mathscr{M}(t, \varphi) * M(t, \varphi)\ .\end{aligned}$$

Apply (3.9) to an element $u \in D_2$, differentiate the resulting identity and then convolute both sides with the Heaviside function $h(t) = 0$ if $t < 0$, $h(t) = 1$ if $t \geqq 0$ (that is, integrate both sides from 0 to t). The final result is, taking into account that $M(0, \varphi) = 0$,

$$\tag{3.10} \begin{aligned}\tilde{T}(t)u = (\varphi T)(t)u &- \Big(\int_0^t (\varphi T)(s)ds\Big) * M'(t, \varphi)u \\ &- \Big(\int_0^t (\varphi T)(s)ds\Big) * \mathscr{M}(t, \varphi) * M'(t, \varphi)u^{(8)}\ .\end{aligned}$$

Differentiating (3.10) once more and observing that $M'(0, \varphi)u = 0$ we obtain

$$\tag{3.11} \begin{aligned}\tilde{T}'(t)u = (\varphi T)'(t)u &- \Big(\int_0^t (\varphi T)(s)ds\Big) * M''(t, \varphi)u \\ &- \Big(\int_0^t (\varphi T)(s)ds\Big) * \mathscr{M}(t, \varphi) * M''(t, \varphi)u\ .\end{aligned}$$

Differentiating still one more time,

$$\tag{3.12} \begin{aligned}\tilde{T}''(t)u = (\varphi T)''(t)u &- (\varphi T)(s) * M''(t, \varphi)u \\ &- (\varphi T)(t) * \mathscr{M}(t, \varphi) * M''(t, \varphi)u\ .\end{aligned}$$

Finally, we modify (3.10) and (3.11) by integrating by parts in their right-hand sides. The result is

$$\tag{3.13} \begin{aligned}\tilde{T}(t)u = (\varphi T)(t)u &- \varphi(t)\int_0^t T(s)ds * M'(t, \varphi)u \\ &+ \int_0^t \varphi'(s)\Big(\int_0^s T(r)dr\Big)ds * M'(t, \varphi)u \\ &- \varphi(t)\int_0^t T(s)ds * \mathscr{M}(t, \varphi) * M'(t, \varphi)u \\ &+ \int_0^t \varphi'(s)\Big(\int_0^s T(r)dr\Big)ds * \mathscr{M}(t, \varphi) * M'(t, \varphi)\ .\end{aligned}$$

(8) In convoluting with the Heaviside function we make use of associativity of the convolution product.

$$
(3.14)\qquad \begin{aligned}
\widetilde{T}'(t)u = (\varphi T)'(t)u &- \varphi(t)\int_0^t T(s)ds * M''(t, \varphi) \\
&+ \int_0^t \varphi'(s)\Big(\int_0^s T(r)dr\Big) * M''(t, \varphi)u \\
&- \varphi(t)\int_0^t T(s)ds * \mathscr{M}(t, \varphi) * M''(t, \varphi)u \\
&+ \int_0^t \varphi'(s)\Big(\int_0^s T(r)dr\Big) * \mathscr{M}(t, \varphi) * M''(t, \varphi)u \ .
\end{aligned}
$$

Applying now Lemma 2.1 and the (already proven) fact that

$$A\int_0^t T(s)ds$$

is a strongly continuous function to (3.13), (3.14) we immediately see that $\widetilde{T}(t)u$, $\widetilde{T}'(t)u \in D(A)$ and that $A\widetilde{T}(\cdot)u$, $A\widetilde{T}'(\cdot)u$ are strongly continuous functions in $[0, \infty[$. Operating in the same way with (3.11), (3.12) we can prove that $\widetilde{T}(t)u$, $\widetilde{T}''(t)u \in D(B)$ and $B\widetilde{T}'(\cdot)u$, $B\widetilde{T}''(\cdot)u$ are as well strongly continuous functions in $[0, \infty[$. Let now

$$u(t) = \widetilde{T}(t)u,\ t \geqq 0,\ u \in D_2 \ .$$

By looking at (3.10), (3.11) we deduce that

$$u(0) = 0,\ u'(0) = u \ .$$

We want to show now that $u(\cdot)$ is a solution of (3.1). Define, for $\bar{\omega}$ large enough, $u \in D$

$$v(t) = \frac{1}{2\pi i}\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty} \frac{e^{\lambda t}}{\lambda^4} R(\lambda)u\,d\lambda \ .$$

As in the proof of Theorem 2.5 it can be shown that v satisfies the equation

$$v''(t) + Bv'(t) + Av(t) = t^3 u \ . \tag{3.15}$$

Since $R(\lambda)u$ is the Laplace transform of $\widetilde{T}(t)u = u(t)$,

$$v(t) = \frac{1}{3!}\int_0^t (t - s)^3 u(t)dt \ .$$

Replacing this expression for $u(\cdot)$ in (3.15) and differentiating three times the resulting identity we obtain

$$u''(t) + Bu'(t) + Au(t) = 0$$

as desired. By differentiating once more we see that $\widetilde{T}'(\cdot)u$, $u \in D_2$ is as well a solution of (3.1) (the fact that $\widetilde{T}'(\cdot) \in C^{(2)}([0, \infty[)$, and that $A\widetilde{T}'(\cdot)u$, $B\bar{T}''(\cdot) \in C^{(0)}([0, \infty[)$ have been already demonstrated);

moreover, it follows from (3.11) that $\tilde{T}'(0)u = u$ and from (3.12) that $\tilde{T}''(0)u = (\varphi T)''(0)u = T''(0)u = -Bu$. If we now define

$$u(t) = \tilde{T}'(t)u + \tilde{T}(t)Bu$$

clearly $u(\cdot)$ is a solution of (3.1) with

$$u(0) = u \quad u'(0) = 0 .$$

Moreover, if follows immediately from (3.3) that the operator $\tilde{S}(t) = \tilde{T}'(t) + \tilde{T}(t)B$ (domain: D) has a bounded extension to all of E (which we design with the same symbol); this extension, as a function of t, is given by

$$\begin{aligned}\tilde{S}(t) &= (\varphi T)'(t) + \overline{(\varphi T)(t)B} \\ &\qquad + \mathscr{N}(t, \varphi) * [(\varphi T)'(t) + \overline{(\varphi T)(t)B}] \\ &= (\delta \otimes I + \mathscr{N}(t, \varphi)) * ((\varphi S)(t) + (\varphi' T)(t)) .\end{aligned}$$

This equation can be used as (2.43) was used in § 2 to show that

$$|\tilde{S}(t)| \leqq Ke^{\omega' t}$$

in $t \geqq 0$.

We have proved at this stage that if u_0, u_1 are arbitrary elements in D_2 then

$$u(t) = \tilde{S}(t)u_0 + \tilde{T}(t)u_1,\ t \geqq 0 , \tag{3.16}$$

is a solution of (3.1) in $t \geqq 0$, with $u(0) = u_0$, $u'(0) = u_1$. The proof of Theorem 3.1 will be ended as soon as we show that *any* solution of (3.1) admits the representation (3.16)-even if u_0, u_1 do not belong to D_2. In order to achieve this we begin by solving the inhomogeneous equation

$$u''(t) + Bu'(t) + Au(t) = f(t) \tag{3.17}$$

in $[0, \infty[$ (as in Lemma 2.2 $f(\cdot)$ belongs to $C^{(1)}([0, \infty[))$. Observe first that it follows from (3.4) (by pre-multiplication by B) that $\tilde{T}(t)E \subseteqq D(B)$ and that $B\tilde{T}(\cdot)$ is an $\mathscr{L}(E)$-valued, strongly continuous function in $[0, \infty[$. It also follows from (3.4)-this time by differentiation-that $\tilde{T}'(\cdot)$ is as well an $\mathscr{L}(E)$-valued, strongly continuous function in $[0, \infty[$. Finally, it can be proved that $\left(\int_0^t \tilde{T}(s)ds\right)E \subseteqq D(A)$ and that

$$A\int_0^t \tilde{T}(s)ds = I - B\tilde{T}(t) - \tilde{T}'(t),\ t \geqq 0 \tag{3.18}$$

much in the same way equality (2.6) was proved. Imitating now the proof of Lemma 2.2 we can use (3.18) to show that

$$u(t) = \int_0^t \tilde{T}(t-s)f(s)ds = \int_0^t \tilde{T}(s)f(t-s)ds \tag{3.19}$$

is a solution of (3.17) in $[0, \infty[$ with $u(0) = u'(0) = 0$. It is the *only* such solution. For, let $v(\cdot)$ be another solution of (3.17) with the same initial conditions. Then $w = u - v$ satisfies (3.1) with initial data $w(0) = w'(0) = 0$. Since the Cauchy problem for (3.1) is well posed in $[0, a]$, $w(t) = 0$ for $0 \leqq t \leqq a$; in particular $w(a) = w'(a) = 0$. Applying the same reasoning to the function $w(t + a)$ we obtain $w(t) = 0$ in $a \leqq t \leqq 2a, \cdots$ etc.

Observe, finally, that it follows from the definitions of $\tilde{S}$, $\tilde{T}$ and from an examination of their Laplace transforms that if $u \in D(A)$,

$$\tilde{S}'(t)u = -\tilde{T}(t)Au \tag{3.20}$$

in $t \geqq 0$.

Let now $u(\cdot)$ be any solution of (3.1). Define

$$u_1(t) = \int_0^t (t-s)(u(s) - u(0))ds , \qquad t \geqq 0 .$$

We have

$$u_1'(t) = \int_0^t (u(s) - u(0))ds = \int_0^t (t-s)u'(s)ds ,$$

$$u_1''(t) = u(t) - u(0) = \int_0^t (t-s)u''(s)ds + tu'(0) .$$

Accordingly, $u_1(\cdot)$ satisfies

$$u_1''(t) + Bu_1(t) + Au_1(t) = tu'(0) - \frac{t^2}{2}Au(0)$$

in $t \geqq 0$. Applying the previous comments on (3.17) and observing that $u_1(0) = u_1'(0) = 0$, we obtain

$$u_1(t) = \int_0^t [(t-s)\tilde{T}(s)u'(0) - \tfrac{1}{2}(t-s)^2\tilde{T}(s)Au(0)]ds .$$

Differentiating twice

$$u(t) = \tilde{T}(t)u'(0) - \int_0^t \tilde{T}(s)Au(0)ds + u(0) , \qquad t \geqq 0$$

which via (3.20) shows that formula (3.16) is valid for any solution of (3.1) in $t \geqq 0$. This clearly implies that the Cauchy problem for (3.1) is well posed in $[0, \infty[$. It has been shown in the course of the proof that both $\tilde{S}(\cdot)$, $\tilde{T}(\cdot)$ increase at most exponentially at infinity, which completes the demonstration of Theorem 3.1.

4. We deal here with conditions on the coefficients A, B, of (1.1) that guarantee that the hypotheses necessary for the proof of Theorems (2.1) and (3.1) are satisfied.

THEOREM 4.1. *Let the Cauchy problem for*

$$u''(t) + Bu'(t) + Au(t) = 0 \tag{4.1}$$

be well posed in $[0, \infty[$. *Assume that, either* (a) B *is bounded* (*hence everywhere defined*) *or* (b) $D_0 = D(A)$ *and* $AD(A) = E$. *Then* $T'(\cdot)$, $BT(\cdot)$ *are* $\mathscr{L}(E)$*-valued, strongly continuous functions in* $[0, \infty[$.

Proof. (a) The assertion about $BT(\cdot)$ is evident. Let $u \in E$,

$$u(t) = \int_0^t (t - s)T(s)u\,ds\ , \quad t \geqq 0\ . \tag{4.2}$$

If $u \in D_1$, $AT(t)u = -BT'(t)u - T''(t)u$; applying this and Lemma 2.0 to (4.2),

$$Au(t) = -T(t)u - B\int_0^t T(s)u\,ds + tu\ . \tag{4.3}$$

Since A is closed, however, it follows from (4.3) that $u(t) \in D(A)$ and that in fact the equality holds for all $u \in E$. Consequently $u(\cdot)$ satisfies

$$u''(t) + Bu'(t) + Au(t) = tu \tag{4.4}$$

in $t \geqq 0$; moreover $u(0) = u'(0) = 0$. Assume now that $u \in D_0$. Then it is easy to see by means of some elementary manipulations that

$$v(t) = \frac{1}{2!}\int_0^t (t - s)^2(S(s)u - T(s)Bu)ds$$

satisfies (4.4) and assumes the same initial values as u. Consequently $u(\cdot) = v(\cdot)$; differentiating twice,

$$T(t)u = \int_0^t (S(s)u - T(s)Bu)ds\ . \tag{4.5}$$

As B is bounded, (4.5) must hold as well for any $u \in E$, which establishes our assertion on $T'(\cdot)$.

(b) Let $u \in K = \{u \in D(A); Au \in D_1\}$. Define

$$u(t) = u - \int_0^t T(s)Au\,ds\ , \qquad t \geqq 0\ .$$

A simple computation shows that $u(\cdot)$ is a solution of (4.1); plainly $u(0) = u$, $u'(0) = 0$. In view of (2.1) $u(t) = S(t)u$, that is

$$S(t)u - u = -\int_0^t T(s)Au\,ds \;. \tag{4.6}$$

Consider now the operator A from $D(A)$ (endowed with the graph norm $|u|_{D(A)} = |u| + |Au|$) to E. Since A is closed $D(A)$ is a Banach space. On the other hand, A is onto, thus by the open mapping principle ([3], Chapter II, p. 55) it transforms open sets into open sets. This plainly implies that K is dense in $D(A)$ (if it were not, there would be an open set Ω in $D(A)$ disjoint from K; then $A\Omega$-which is open-would be disjoint from D_1, absurd in view of the density of D_1). Let now $u \in D(A)$, $\{u_n\}$ a sequence in K such that $u_n \to u$ in $D(A)$. Writing (4.6) for u_n and then letting $n \to \infty$ we see that it holds for any $u \in D(A)$; since $S(\cdot)u \in C^2([0, \infty[)$ for those u, $T(\cdot)Au \in C^{(1)}([0, \infty[)$. Since any element of E can be written in the form Au, the assertion on $T'(\cdot)$ follows.

REMARK 4.2. Theorem 4.1 (a) together with Theorem 2.1 furnishes a new proof of the exponential increase of the solutions of the equation $u'' + Au = 0$ (see [8], p. 9 and [4], part I, p. 90).

REMARK 4.3. Under the hypotheses in (b) a number of additional properties of the propagators can be established. For instance, it follows from (2.51) and from the fact that $S(t)D(A) \subseteqq D(A)$ for all $t \geqq 0$ (consequence of the definition of $S(t)$) that $T(s)T(t)E \subseteqq D(A)$ for all $s, t \geqq 0$ and that $AT(s)T(t)$ is an $\mathscr{L}(E)$-valued function, strongly continuous jointly in both variables in $[0, \infty[\times [0, \infty[$. Assume, to simplify, that A is in addition, one-to-one and thus has a bounded inverse A^{-1}. Then we can write

$$AT(s)T(t) = AS(s)S(t)A^{-1} - AS(s+t)A^{-1}\,, \qquad s, t \geqq 0\,.$$

Similarly, we can combine the equality

$$AS(t)A^{-1} = -BS'(t)A^{-1}u - S''(t)A^{-1}u$$

with the expressions obtained differentiating (4.6) once and twice respectively, to obtain

$$AS(t)A^{-1} = BT(t) + T'(t)\,.$$

REMARK 4.4. All the results in this section have analogues for the case in which the Cauchy problem for (4.1) is well posed in a *finite* interval. The proofs are identical.

5. We present here several counter-examples that illuminate the rôle of the hypotheses in Theorems 2.1 and 3.1. Throughout this section E will be a separable Hilbert space, $\{\varphi_n\}$, $1 \leqq n < \infty$ a fixed

complete orthonormal system in E. The operators A, B are given by

(5.1) $$A\varphi_n = a_n\varphi_n \quad B\varphi_n = b_n\varphi_n \text{[9]}$$

where the (complex) coefficients a_n, b_n, $n \geqq 1$ will be chosen in each case such as to produce the desired effect. (Observe incidentally that A and B are normal operators commuting with each other for any choice of $\{a_n\}$, $\{b_n\}$.)

Consider the Cauchy problem for

(5.2) $$u''(t) + Bu'(t) + Au(t) = 0$$

(in $[0, a]$ for $a < \infty$ or in $[0, \infty[$). If $u(t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n$ is a solution of (5.1) then it is plain that each coordinate $u_n(\cdot)$ must satisfy the scalar equation

$$u_n''(t) + b_n u_n'(t) + a_n u_n(t) = 0$$

$n = 1, 2, \cdots$ with initial conditions

$$u_n(0) = u_{0,n} \quad u_n'(0) = u_{1,n}$$

where

$$u(0) = u_0 = \sum_{n=1}^{\infty} u_{0,n}\varphi_n \quad u'(0) = u_1 = \sum_{n=0}^{\infty} u_{1,n}\varphi_n \,.$$

This makes clear that the propagators S, T must be defined by

(5.3) $$S(t)\varphi_n = \frac{\lambda_n e^{\mu_n t} - \mu_n e^{\lambda_n t}}{\lambda_n - \mu_n}\varphi_n$$

(5.4) $$T(t)\varphi_n = \frac{e^{\lambda_n t} - e^{\mu_n t}}{\lambda_n - \mu_n}\varphi_n$$

where λ_n, μ_n are the roots of the n-th "characteristic polynomial"

(5.5) $$\lambda^2 + \lambda b_n + a_n = 0 \,,$$

$n = 1, 2, \cdots$ (if $\lambda_n = \mu_n$ we must, of course modify (5.3), (5.4) but we shall not encounter this case in our examples). Accordingly, we see that a necessary condition for the Cauchy problem for (5.2) to be well posed in $[0, a]$ (resp. in $[0, \infty[$) is that the functions

(5.6) $$\sigma(t) = |S(t)| = \sup_{n \geqq 1} \left| \frac{\lambda_n e^{\mu_n t} - \mu_n e^{\lambda_n t}}{\lambda_n - \mu_n} \right|$$

(5.7) $$\tau(t) = |T(t)| = \sup_{n \geqq 1} \left| \frac{e^{\lambda_n t} - e^{\mu_n t}}{\lambda_n - \mu_n} \right|$$

[9] That is, $D(A) = \{u \in E;\ \sum |a_n(u, \varphi_n)|^2 < \infty\}$, $Au = \sum a_n(u, \varphi_n)\varphi_n$, where $(\cdot, \cdot)$ denotes the scalar product in E. Same observation about B.

should be bounded on $[0, a]$ (resp. on compacts of $[0, \infty[$).[10] Conversely, the preceding conditions imply that the Cauchy problem is well posed; for if, say the Fourier coefficients of u_0, u_1 are all zero except for a finite number then $S(t)u_0 + T(t)u_1$ furnishes a solution of (5.2) in $t \geqq 0$ with $u(0) = u_0$, $u'(0) = u_1$. Moreover, it follows from the preceding considerations (that is, taking coordinates) that *any* solution $u(\cdot)$ of (5.2) must be of the form $u(t) = S(t)u(0) + T(t)u'(0)$; then

$$|u(s)| \leqq \left(\sup_{0 \leqq s \leqq t} \sigma(s)\right)|u(0)| + \left(\sup_{0 \leqq s \leqq t} \tau(s)\right)|u'(0)| .$$

Our first result is

THEOREM 5.1. *Let $a > 0$. Then there exist A, B (of the form* (5.1)) *such that the Cauchy problem for* (5.2) *is well posed in* $[0, a]$*-but not well posed in any interval of the form* $[0, a']$, $a' > a$.

Proof. We set $A = 0$ (that is $a_n = \mu_n = 0$); as for the coefficients of B in (5.1), we set

$$\lambda_n = \frac{1}{a}\log n + \frac{i}{a}(n^2 - (\log n)^2)^{1/2} , \qquad n \geqq 1 \tag{5.8}$$

(by (5.5), $b_n = -\lambda_n$). As $S(t) \equiv I$, we only have to check the boundedness (or unboundedness) of $\tau(\cdot)$ in (5.7). But

$$\begin{aligned} a(n^{(t-a)/a} - n^{-1}) &\leqq \left|\frac{e^{\lambda_n t} - e^{\mu_n t}}{\lambda_n - \mu_n}\right| \\ &\leqq a(n^{(t-a)/a} + n^{-1}) , \qquad t \geqq 0 . \end{aligned} \tag{5.9}$$

Consequently $\tau(t) \leqq 2a$ if $t \leqq a$, $\tau(t) = \infty$ if $t > a$. This establishes the required result.

REMARK 5.2. It is quite simple to see why Theorem 3.1 fails to apply to the preceding example. In fact, it follows from (5.5) that in our case

$$T'(t)\varphi_n = e^{\lambda_n t}\varphi_n ,$$

$n \geqq 1$; but, as $|e^{\lambda_n t}| = n^{t/a}$, $T'(t)$ is *not* a bounded operator for any t-except of course $t = 0$. We when also note that we have $D_0 = D(A) = E$; but, since $AD(A) = \{0\} \neq E$, Proposition 4.1 also fails to apply. We introduce now a slight modification in the example. Set λ_n as in (5.8) but set now $\mu_n = \mu$, $n \geqq 1$ where $\mu \neq \lambda_n$ for all $n \geqq 1$,

[10] An operator of the form $Qu = \sum q_n(u, \varphi_n)\varphi_n$ in E is bounded if and only if $q = \sup |q_n| < \infty$ (moreover, $q = |Q|$).

$\mu \neq 0$. Since $a_n = \mu\lambda_n$, $|a_n| \neq 0$, $|a_n| \to \infty$, A has a bounded inverse, in particular $AD(A) = E$. However, it is not difficult to see that $\sigma(t) = \tau(t) = \infty$ for $t > a$, σ and τ are bounded in $[0, a]$. This shows that none of the two hypotheses in Proposition 4.1 (b) can be altogether discarded.

We now show that, by judicious choice of A, B in (5.2) the propagators can be made to increase as fast as desired even if the Cauchy problem for (5.2) is well posed in $[0, \infty[$.

THEOREM 5.3. *Let $\omega(\cdot)$ be an arbitrary function in $[0, \infty]$, bounded on compact subsets therein. Then there exist A, B (of the form* (5.1)) *such that* (a) *The Cauchy problem for* (5.2) *is well posed in* $[0, \infty[$. (b)

$$|S(t)| \geqq \omega(t),\ |T(t)| \geqq \omega(t) \tag{5.10}$$

for $t \geqq 1$.

Proof. Let $\Omega = \{\omega_n\}$, $n \geqq 1$ be a sequence of positive numbers such that

$$2 \leqq \omega_1 \leqq \omega_2^{1/2} \leqq \omega_3^{1/3} \leqq \cdots,\ \lim_{n\to\infty} \omega_n^{1/n} = \infty \tag{5.11}$$

but otherwise arbitrary. Define

$$\begin{aligned} \alpha_n &= \omega_n^{1/n} \exp\left(\frac{1}{n}\omega_n^{1/n}\right), \\ \beta_n &= \exp \omega_n^{1/n} \end{aligned} \tag{5.12}$$

for $n \geqq 1$, and let

$$m(t) = \sup_{n\geqq 1} \frac{\alpha_n^t}{\beta_n}, \qquad t \geqq 0\,. \tag{5.13}$$

Noting that

$$\begin{aligned} \frac{\alpha_n^t}{\beta_n} &= \frac{\omega_n^{t/n} \exp\left(\dfrac{t}{n}\omega_n^{1/n}\right)}{\exp \omega_n^{1/n}} \\ &= \frac{(\omega_n^{1/n})^t}{\exp\left(\left(1 - \dfrac{t}{n}\right)\omega_n^{1/n}\right)} \leqq \frac{(\omega_n^{1/n})^t}{\exp\left(\dfrac{1}{2}\omega_n^{1/n}\right)} \end{aligned}$$

for $1 - t/n \geqq 1/2$ we see that $\alpha_n^t = o(\beta_n)$ as $n \to \infty$ for all t; then

$m(t) < \infty$ for all $t \geqq 0$. Moreover, for each t there exists an integer $n = n(t)$[11] such that

$$m(t) = \frac{\alpha_n^t}{\beta_n} .$$

Let now $t < t'$; since $\alpha_n > 1$ for all n,

$$m(t) = \frac{\alpha_{n(t)}^t}{\beta_{n(t)}} < \frac{\alpha_{n(t)}^{t'}}{\beta_{n(t)}} \leqq m(t')$$

accordingly the function $m(\cdot)$ is increasing in $[0, \infty[$, thus bounded on compacts therein. Also,

$$m(n) \geqq \frac{\alpha_n^n}{\beta_n} = \omega_n , \quad n \geqq 1 . \tag{5.14}$$

Define now

$$\gamma_n = \log \alpha_n = \log \omega_n^{1/n} + \frac{1}{n}\omega_n^{1/n} .$$

In view of the inequality $\log x + x \leqq e^x/\sqrt{2}$ valid (at least) for $x \geqq 2$ and of the fact that $\omega_n^{1/n} \geqq 2$, we have

$$\gamma_n \leqq \frac{1}{\sqrt{2}}\beta_n . \tag{5.15}$$

We now choose a_n, b_n in (5.1)-or, what is the same, λ_n, μ_n, the roots of (5.5), in the following way:

$$\lambda_n = \gamma_n + i(\beta_n^2 - \gamma_n^2)^{1/2},\ \mu_n = 1 , \qquad n \geqq 1 .$$

Plainly $|\lambda_n| = \beta_n \geqq e^2$; on the other hand, by virtue of (5.15)

$$(\beta_n^2 - \gamma_n^2)^{1/2} \geqq \gamma_n ,$$

thus the sequence $\Lambda = \{\lambda_n\}$ is contained in the region

$$|\lambda| \geqq e^2,\ 0 \leqq \operatorname{Re}\lambda \leqq \operatorname{Im}\lambda .$$

Accordingly there exist constants $\Theta > \theta > 0$ *independent of* Ω such that

$$\theta \leqq \frac{|\lambda_n|}{|\lambda_n - 1|} \leqq \Theta , \qquad n \geqq 1 . \tag{5.16}$$

We calculate now the functions σ, τ in (5.6), (5.7). With the foregoing choice of λ_n, μ_n we plainly have

[11] Not necessarily unique.

$$\frac{|\lambda_n|}{|\lambda_n - 1|}\left(\frac{|e^{\lambda_n t}|}{|\lambda_n|} - e^t\right) \leqq \left|\frac{\mu_n e^{\lambda_n t} - \lambda_n e^{\mu_n t}}{\lambda_n - \mu_n}\right|$$
$$\leqq \frac{|\lambda_n|}{|\lambda_n - 1|}\left(\frac{|e^{\lambda_n t}|}{|\lambda_n|} + e^t\right),$$

$$\frac{|\lambda_n|}{|\lambda_n - 1|}\left(\frac{|e^{\lambda_n t}|}{|\lambda_n|} - \frac{e^t}{|\lambda_n|}\right) \leqq \left|\frac{e^{\lambda_n t} - e^{\mu_n t}}{\lambda_n - \mu_n}\right|$$
$$\leqq \frac{|\lambda_n|}{|\lambda_n - 1|}\left(\frac{|e^{\lambda_n t}|}{|\lambda_n|} + \frac{e^t}{|\lambda_n|}\right).$$

In view of (5.16), we obtain

$$\theta(m(t) - e^t) \leqq \sigma(t) \leqq \Theta(m(t) + e^t) \tag{5.17}$$

$$\theta(m(t) - e^t) \leqq \tau(t) \leqq \Theta(m(t) + e^t) \tag{5.18}$$

in $t \geqq 0$. The inequalities in the right-hand sides of (5.17), (5.18) imply that the Cauchy problem for (5.2) is well posed in $[0, \infty[$. It only remains to choose the sequence Ω in such a way that the inequalities (5.10) are satisfied. Observe first that we may assume, without loss of generality, that $\omega(\cdot)$ is nondecreasing. Define

$$\omega_n = \left(\frac{\omega(n+1)}{\theta} + e^n\right)^n, \qquad n \geqq 1. \tag{5.19}$$

A moment's observation shows that $\{\omega_n\}$ satisfies all the required conditions. Let now $t \geqq 1$, $n = [t]$, the greatest integer $\leqq t$. Taking into account (5.17) and (5.14) we obtain

$$\begin{aligned}\sigma(t) &\geqq \theta(m(t) - e^t) \geqq \theta(m(n) - e^n)\\ &\geqq \theta(\omega_n - e^n) \geqq \theta(\omega_n^{1/n} - e^n) = \omega(n+1) \geqq \omega(t).\end{aligned}$$

The corresponding inequality for τ is obtained in exactly the same way.

REMARK 5.4. In the preceding example we have $a_n = \lambda_n \mu_n = \lambda_n$, $b_n = -(\lambda_n + \mu_n) = -(\lambda_n + 1)$ thus Equation (5.3) has the special form

$$u''(t) - (A + I)u'(t) + Au(t).$$

The operator A has a bounded inverse -then $AD(A) = E$- but $D_0 \neq D(A)$. It is not difficult to see that the Cauchy problem for

$$u''(t) - Au'(t) = 0 \tag{5.20}$$

is also well posed in $[0, \infty[$; now $S(t) \equiv I$, $D_0 = E$, but

$$\tau(t) = |T(t)| \geqq (m(t) - 1)$$

and thus it can be forced to increase as rapidly as one wishes.

6. The results in §'s 2 and 3 can be generalized -at the price of some complication in the notations but with essentially the same ideas- to equations of order n. We sketch here the proofs of these generalizations. The equation in question is now

$$u^{(n)}(t) + \sum_{k=0}^{n-1} A_k u^{(k)}(t) = 0 \tag{6.1}$$

where $A_0, \cdots, A_{n-1}$ are closed, densely defined operators in E. A solution of (6.1) -say, in $[0, \infty[$- is a function $u(\cdot) \in C^{(n)}([0, \infty[)$ such that $u^{(k)}(t) \in D(A_k)$, $A_k u^{(k)}(\cdot) \in C^{(0)}([0, \infty[)$, $0 \leqq k \leqq n-1$, and (6.1) is satisfied everywhere. The Cauchy problem for (6.1) is well posed in $[0, \infty[$ if and only if

(a) There exist dense subspaces $D_0, \cdots, D_{n-1}$ of E such that for every $u_0 \in D_0, \cdots, u_{n-1} \in D_{n-1}$ there is a solution $u(\cdot)$ of (6.1) in $[0, \infty[$ with $u^{(k)}(0) = u_k$, $0 \leqq k \leqq n-1$.

(b) For every $t \geqq 0$ there exist constants $K_0(t), \cdots, K_{n-1}(t) < \infty$ such that, for every solution $u(\cdot)$ of (6.1), $0 \leqq s \leqq t$

$$|u(s)| \leqq \sum_{k=0}^{n-1} K_k(t) \, |u^{(k)}(0)| \, .$$

The formulation of the Cauchy problem for a finite interval $[0, a]$ is similar (see § 3 for the case $n = 2$) and is therefore omitted. We now have n propagators $S_0, \cdots, S_{n-1}$; S_k is defined in D_k by

$$S_k(t)u = u_k(t)$$

where $u_k(\cdot)$ is the solution of (6.1) with $u_k^{(l)}(0) = \delta_{kl} u$, δ_{kl} the Kronecker delta, $(0 \leqq k, l \leqq n-1)$ and extended to all of E by continuity. Just as in the case $n = 2$ it can be proved that $S_0, \cdots, S_{n-1}$ are $\mathscr{L}(E)$-valued strongly continuous functions and that if $u(\cdot)$ is any solution of (6.1) in $[0, \infty[$ then

$$u(t) = \sum_{k=0}^{n-1} S_k(t) u^{(k)}(0) \, . \tag{6.2}$$

THEOREM 6.1. *Let the Cauchy problem for* (6.1) *be well posed in* $[0, \infty[$. *Assume that for every* $u \in E$,

$$S_k(\cdot)u \in C^{(k)}([0, \infty[), \; S_{n-1}^{(k-1)}(t)u \in D(A_k) \text{ and } A_k S_{n-1}^{(k-1)}(\cdot)u$$

is continuous in $[0, \infty[, 1 \leqq k \leqq n-1$. *Then there exist constants* $K_0, \cdots, K_{n-1}, \omega_0, \cdots, \omega_{n-1} < \infty$ *such that*

$$|u(t)| \leqq \sum_{k=0}^{n-1} K_k e^{\omega_k t} \, |u^{(k)}(0)| \, , \quad t \geq 0 \, . \tag{6.3}$$

The proof of Theorem 6.1 can be carried out in a series of steps imitating those in the proof of Theorem 2.1. They are as follows.

LEMMA 6.2. *Let* $f(\cdot)$ *belong to* $C^{(1)}([0, \infty[)$. *Then* (a)

$$u(t) = \int_0^t S_{n-1}(t-s)f(s)ds \tag{6.4}$$

is a solution of

$$u^{(n)}(t) + \sum_{k=0}^{n-1} A_k u^{(k)}(t) = f(t) \tag{6.5}$$

with $u(0) = \cdots = u^{(n-1)}(0) = 0$. (b) *If* $v(\cdot)$ *is any other solution of* (6.5) *then*

$$v(t) = \sum_{k=0}^{n-1} S_k(t)v^{(k)}(0) + u(t) \tag{6.6}$$

$u(\cdot)$ *defined by* (6.4).

As in Lemma 2.2 we begin by observing that, if $u \in D_{n-1}$

$$A_0 S_{n-1}(t)u = -\sum_{k=1}^{n-1} A_k S_{n-1}^{(k)}(t)u - S_{n-1}^{(n)}(t)u .$$

Integrating,

$$A_0 \int_0^t S_{n-1}(s)u = -\sum_{k-1}^{n-1} A_k S_{n-1}^{(k-1)}(t)u - S_{n-1}^{(n-1)}(t)u + u .$$

This, and the hypotheses in Theorem 6.1 imply that

$$\left(\int_0^t S_{n-1}(s)ds\right)E \subseteqq D(A_0)$$

and that

$$\begin{aligned} U(t) = A_0 \int_0^t S_{n-1}(s)ds &+ \sum_{k=1}^{n-1} A_k S_{n-1}^{(k-1)}(t) \\ &+ S_{n-1}^{(n-1)}(t) = I , \qquad t \geqq 0 . \end{aligned}$$

Writing, by integration by parts

$$u(t) = \int_0^t S_{n-1}(s)f(0)ds + \int_0^t \left(\int_0^{t-s} S_{n-1}(r)dr\right)f'(s)ds$$

and differentiating, we obtain

$$u^{(k)}(t) = S_{n-1}^{(k-1)}(t)f(0) + \int_0^t S_{n-1}^{(k-1)}(t-s)f'(s)ds ,$$

$1 \leqq k \leqq n$. Finally, inserting $u(\cdot)$ in (6.5) the right-hand side equals

$$U(t)f(0) + \int_0^t U(t-s)f'(s)ds$$

as claimed. Part (b) follows exactly as for second-order equations.

The result of Lemma 6.2 can be used as their analogues in § 2 to prove

LEMMA 6.3. (a) *Let* $u \in D(A_0)$. *Then*

$$S_0'(t)u = -S_{n-1}(t)A_0 u \,. \tag{6.7}$$

(b) *Let* $u \in D_{k-1} \cap D(A_k)$, $1 \leqq k \leqq n-1$. *Then*

$$S_k'(t)u = S_{k-1}(t)u - S_{n-1}(t)A_k u \,. \tag{6.8}$$

COROLLARY 6.4. (a) $D_0 = D(A_0)$. (b) $\bigcap_{k=0}^{n-1} D(A_k)$ *is dense in* E. (c) $D_1 \supseteqq D(A_0) \cap D(A_1)$, $D_2 \supseteqq D(A_0) \cap D(A_1) \cap D(A_2)$, $\cdots$, $D_{n-1} \supseteqq D = \bigcap_{k=0}^{n-1} D(A_k)$. (d) *If* $u \in D$,

$$S_{n-1}^{(n)}(t)u + \sum_{k=0}^{n-1} S_{n-1}^{(k)}(t)A_k u = 0 \,. \tag{6.9}$$

LEMMA 6.5. (a) $P(\lambda) = \lambda^n + \sum_{k=0}^{n-1} \lambda^k A_k$ *is pre-closed for all* λ. (b) *There exist constants* $\alpha, \beta \geqq 0$ *such that* $P(\lambda)$ *is one-to-one for*

$$\operatorname{Re}\lambda \geqq \alpha + \beta \log\left(\sum_{k=0}^{n-1} |\lambda|^k\right) . \tag{6.10}$$

The operator $R(\lambda, \varphi)$ of § 2 is defined here by means of the formula

$$R(\lambda, \varphi)u = \int_0^\infty e^{-\lambda t}\varphi(t)S_{n-1}(t)u\,dt \tag{6.11}$$

where φ is now a n-times continuously differentiable function with compact support and such that $\varphi(0) = 1$. It can be easily seen that $R(\lambda; \varphi)E \subseteqq D$; moreover, if $u \in E$,

$$\begin{aligned} P(\lambda)R(\lambda, \varphi)u &= u + \int_0^\infty e^{-\lambda t} M(t, \varphi)u\,dt \\ &= u + \hat{M}(\lambda, \varphi)u \end{aligned} \tag{6.12}$$

where now

$$\begin{aligned} M(t, \varphi) &= \sum_{j=0}^{n-1} \binom{n}{j} \varphi^{(n-j)}(t) S_{n-1}^{(j)}(t) \\ &\quad + \sum_{k=1}^{n-1} \sum_{j=1}^{k-1} \binom{k}{j} \varphi^{(k-j)} A_k S_{n-1}^{(j)}(t) \,. \end{aligned} \tag{6.13}$$

Observe that if $j \leqq k-1$ (as in 6.13)

$$S_{n-1}^{(j)}(t)u = \frac{1}{(k-j-2)!}\int_0^t (t-s)^{k-j-2} S_{n-1}^{(k-1)}(s)u\,ds$$

then it follows from the hypotheses in Theorem 6.1 that $A_k S_{n-1}^{(j)}(\cdot)$ is a $\mathscr{L}(E)$ -valued, strongly continuous functions- thus the same is true of $M(t, \varphi)$. The operator $R(\lambda)$ is defined again as

$$R(\lambda) = R(\lambda, \varphi)(I + \hat{M}(\lambda, \varphi))^{-1} \tag{6.14}$$

for $\operatorname{Re}\lambda \geqq \omega$, ω such that (2.18) is true. Once proved- as in § 2, but now making use of Lemma 6.5- that $R(\lambda) = P(\lambda)^{-1}$ in $\operatorname{Re}\lambda \geqq \omega$ we construct the "right-handed" representation of $R(\lambda)$, namely

$$R(\lambda) = (I + \hat{N}(\lambda, \varphi))^{-1} R(\lambda; \varphi)\ . \tag{6.15}$$

Here $\hat{N}(\lambda, \varphi)$ is the Laplace transform of $N(t, \varphi)$; $N(t, \varphi)$ is defined by the formula

$$\begin{aligned} N(t, \varphi)u &= \sum_{j=0}^{n-1} \binom{n}{j} \varphi^{(n-j)}(t) S_{n-1}^{(j)}(t)u \\ &\quad + \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \binom{k}{j} \varphi^{(k-j)}(t) S_{n-1}^{(j)}(t) A_k u \end{aligned}$$

for $u \in D = \bigcap_{j=0}^{n-1} D(A_k)$. To show—as we must—that $N(\cdot, \varphi)$ has a $\mathscr{L}(E)$- valued, strongly continuous extension to all of E we go back to the identities (6.8). According to them $S_{n-1}^{(j)}(t)A_k$, $j \leqq k-1$ (say, with domain D) has a bounded extension to all of E (namely,

$$\overline{S_{n-1}^{(j)}(t)A_k} = S_{k-1}^{(j)}(t) - S_k^{(j+1)}(t))\ ,$$

which implies the desired property of $N(\cdot, \varphi)$. Then functions $\mathscr{M}$ and $\mathscr{N}$ are defined in the same way as in § 2, that is $\mathscr{M}(t, \varphi) = \sum_{n=1}^{\infty}(-1)^n M(t, \varphi)^{*n}$, $\mathscr{N}(t, \varphi) = \sum_{n=1}^{\infty}(-1)^n N(t, \varphi)^{*n}$; they are $\mathscr{L}(E)$-valued strongly continuous functions satisfying

$$|\mathscr{M}(t, \varphi)| \leqq K' e^{\omega' t},\ |\mathscr{N}(t, \varphi)| \leqq K e^{\omega t}\ , \tag{6.16}$$

$t \geqq 0$, for convenient constants K, K', ω, ω'. The propagator S_{n-1} satisfies

$$S_{n-1}(t) = (\delta \otimes I + \mathscr{N}(t, \varphi)) * (\varphi S_{n-1})(t) \tag{6.17}$$

and

$$S_{n-1}(t) = (\varphi S_{n-1})(t) * (\delta \otimes I + \mathscr{M}(t, \varphi))\ . \tag{6.18}$$

Equation (6.16) can be used, together with Lemma 6.3 (b) to reconstruct the propagators $S_0, S_1, \cdots, S_{n-2}$ from their values in the support of φ. In fact, we get from Lemma 6.3 (b) that

$$S_k(t) = S_{n-1}^{(n-k-1)}(t) + \sum_{j=k+1}^{n-1} \overline{S_{n-1}^{(j-k-1)}(t)A_j}\ ,$$

$0 \leqq k \leqq n-1$. Consequently,

$$S_k(t) = (\delta \otimes I + \mathscr{N}(t, \varphi)) * \Big((\varphi S_{n-1})^{(n-k-1)} + \sum_{j=k+1}^{n-1} \overline{(\varphi S_{n-1})^{(j-k-1)} A_k} \Big)\ , \tag{6.19}$$

$0 \leqq k < n-1$. Equations (6.17)-(6.19) are used to prove that $S_0, \cdots, S_{n-1}$ increase at most exponentially at ∞. This proves Theorem (6.1). Observe that, using (6.17) the same property can be proved of $\overline{S_{n-1}^{(j)}(t)A_k}$, $j \leqq k-1$; on the other hand, (6.18) can be used to show exponential increase of $A_k S_{n-1}^{(j)}(t)$, $j \leqq k-1$.

The results in § 3 can be generalized-essentially with the same methods used to generalize the results in § 2-to equation 6.1. We limit ourselves to state the following.

THEOREM 6.6. *Let the Cauchy problem for* (6.1) *be well posed in* $[0, a]$. *Assume that* (a) D_2^* *is dense in* E. (b) *For every* $u \in E$,

$$S_k(\cdot)u \in C^{(k)}([0, a]),\ S_{n-1}^{(k-1)}(t)u \in D(A_k)$$

and $A_k S_{n-1}^{(k-1)}(\cdot)u \in C^{(0)}([0, a])$, $1 \leqq k \leqq n-1$. *Then the Cauchy problem for* (6.1) *is actually well posed in* $[0, \infty[$.

Here we have set $D_2^* = \{u \in D;\ A_k u \in D,\ 1 \leqq k \leqq n-1\}$.

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