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BEHAVIOR OF GREEN LINES AT THE KURAMOCHI BOUNDARY OF A RIEMANN SURFACE

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We shall establish necessary and sufficient conditions, in terms of Green lines, for a point of the Kuramochi boundary Γ^k of a hyperbolic Riemann surface R to be of positive harmonic measure.

Explicitly, let $\mathfrak{B}$ be the bundle of all Green lines l issuing from a fixed point of R. It forms a measure space with the Green measure. We call a subset $\mathfrak{A}$ of $\mathfrak{B}$ a distinguished bundle if it has positive measure and there exists a point p in Γ^k such that almost every l in $\mathfrak{A}$ terminates at p. The point p will be referred to as the end of $\mathfrak{A}$.

Our main result is that a point p of Γ^k has positive measure if and only if there exists a distinguished bundle $\mathfrak{A}$ whose end is p.

We shall also give an intrinsic characterization of the latter property, without reference to points of Γ^k: A bundle $\mathfrak{A}$ is distinguished if and only if it has positive measure and for every HD-function u there exists a real number c_u such that u has the limit c_u along almost every l in $\mathfrak{A}$.

1. Green lines

1. Let R be a hyperbolic Riemann surface, the hyperbolicity characterized by the existence of Green's functions. Fix a point $z_0 \in R$ and denote by $g(z) = g(z, z_0)$ the Green's function on R with singularity z_0. Consider the differential equations

$$\frac{dr(z)}{r(z)} = -dg(z), \qquad r(z_0) = 0 , \tag{1}$$

$$d\theta(z) = -*dg(z) . \tag{2}$$

Equation (1) has the unique solution $r(z) = e^{-g(z)}$ on R with $0 \leqq r(z) < 1$. In any simply connected subregion of $R - z_0$ where $dg(z) \neq 0$, equation (2) also has a solution $\theta(z)$, unique up to an additive constant. The global solution $\theta(z)$, however, is a multivalued harmonic function.

Set $G_\rho = \{z \in R \mid r(z) < \rho\}$, $C_\rho = \partial G_\rho (0 < \rho < 1)$. For a sufficiently small ρ, the analytic function $w = \varphi(z) = r(z)e^{i\theta(z)}$ is single-valued and gives a univalent conformal mapping of G_ρ onto the disk $|w| < \rho$. Denote by ρ_0 the supremum of all ρ with this property.

2. An open arc α is called a *Green arc* if $dg(z) \neq 0$ for all $z \in \alpha$,

and a branch of θ is constant on α. The set of Green arcs is partially ordered by inclusion. A maximal Green arc in this partially ordered set is called a *Green line.*

A Green line l is said to *issue from* z_0 if $z_0 \in \bar{l}$. We denote by $\mathfrak{B}$ the set of Green lines issuing from z_0 and use the suggestive term *bundle* for a subset $\mathfrak{A}$ of $\mathfrak{B}$, with the case $\mathfrak{A} = \mathfrak{B}$ not excluded.

For a fixed $\rho \in (0, \rho_0)$ and a given $p \in C_\rho$ let $l(p)$ be the Green line in $\mathfrak{B}$ passing through p. Making use of the function $w = \varphi(z) = r(z)e^{i\theta(z)}$ we see that the mapping $p \to l(p)$ is bijective; let $p(l)$ be the inverse mapping. We call a bundle $\mathfrak{A} \subset \mathfrak{B}$ *measurable* if $p(\mathfrak{A})$ is measurable in C_ρ, and define the *Green measure* of $\mathfrak{A}$ by

$$m(\mathfrak{A}) = \frac{1}{2\pi}\int_{p(\mathfrak{A})} d\theta(z) = -\frac{1}{2\pi}\int_{p(\mathfrak{A})} *dg(z) \ . \tag{3}$$

The space $(\mathfrak{B}, m)$ is a probability space, i.e., a measure space of total measure unity. The definition is independent of the choice of $\rho \in (0, \rho_0)$.

3. Fix an $l \in \mathfrak{B}$. The number $a(l) = \sup_{z \in l} r(z)$ is in $(0, 1]$. If $a(l) < 1$, then l terminates at a point of R at which $dg = 0$. Such an l is called *singular.* If $a(l) = 1$, then l tends to the ideal boundary of R and is called *regular.* The bundle $\mathfrak{B}_r$ of regular Green lines "almost" comprises $\mathfrak{B}$, that is, $m(\mathfrak{B}_r) = 1$. This is a result of Brelot-Choquet [1] (cf. [7], [8]).

2. Compactifications.

4. Let R^c be a compactification of R, i.e., a compact Hausdorff space containing R as its open dense subspace. For a bounded continuous function φ on the ideal boundary $\Gamma^c = R^c - R$ of R, denote by $U_\varphi^{R^c}$ the class of superharmonic functions s on R such that

$$\liminf_{z \in R, z \to p} s(z) \geqq \varphi(p)$$

for every $p \in \Gamma^c$. The function

$$H_\varphi^{R^c}(z) = \inf_{s \in U_\varphi^{R^c}} s(z)$$

is harmonic on R. We assume that R^c is a *resolutive* compactification (cf. Constantinescu-Cornea [2]), that is, $\varphi \to H_\varphi^{R^c}(z)$ is a continuous linear functional. Then for $z_0 \in R$ there exists a measure μ^c, called the *harmonic measure on* Γ^c, and a function $P^c(z, p)$ on $R \times \Gamma^c$ with properties $P_c(z_0, p) \equiv 1$,

$$H_\varphi^{R^c}(z) = \int_{\Gamma^c} P^c(z, p)\varphi(p)d\mu^c(p) \ . \tag{4}$$

This representation extends to bounded Borel measurable functions φ on Γ^c.

Let $\widetilde{HD}(R)$ be the class of harmonic functions $u \geqq 0$ on R such that there exists a decreasing sequence $\{u_n\} \subset HD(R)$ with $u = \lim_n u_n$ on R. A function $u \in \widetilde{HD}(R)$ is said to be *$\widetilde{HD}$-minimal* if for every $v \in \widetilde{HD}(R)$ with $v \leqq u$ on R there exists a constant c_v such that $v = c_v u$ on R. We shall call the compactification R^c *$\widetilde{HD}$-compatible* if the following condition is satisfied: $u \in \widetilde{HD}(R)$ is $\widetilde{HD}$-minimal if and only if there exists a point $p_0 \in \Gamma^c$ with $\mu^c(p_0) > 0$ and a number $k > 0$ such that

$$u(z) = k \int_{p_0} P^c(z, p) d\mu^c(p) . \tag{5}$$

5. The *Royden compactification* R^* of R, with the *Royden boundary* $\Gamma = R^* - R$, is a typical example of an $\widetilde{HD}$-compatible compactification (see [6], [8]). We let μ and P stand for μ^c and P^c corresponding to R^*.

A compactification R^c is said to *lie below* R^* if there exists a continuous mapping $\pi = \pi^c$ of R^* onto R^c such that $\pi | R$ is the identity and $\pi^{-1}(R) = R$. Clearly π is unique and we have

$$\int_{\Gamma^c} P^c(z, p)\varphi(p) d\mu^c(p) = \int_{\Gamma} P(z, p^*)\varphi(\pi(p^*)) d\mu(p^*) \tag{6}$$

for every bounded Borel function φ on Γ^c.

6. We are interested in the behavior of $l \in \mathfrak{B}_r$ in R^c. We set

$$e^c(l) = \bar{l}^c - l \cup \{z_0\} , \tag{7}$$

with $\bar{l}^c$ the closure of l in R^c, and call $e^c(l)$ the *end part* of l in R^c. It is a compact set in Γ^c. If

$$\mathfrak{B}^c = \{l \in \mathfrak{B}_r \,|\, e^c(l) \text{ is a single point}\}$$

is of measure $m(\mathfrak{B}^c) = 1$, then we call R^c *Green-compatible.*

We shall make use of a result of Maeda [4]: A metrizable compactification R^c which lies below R^* is Green-compatible.

7. A compactification R^c of R is said to be *of type G* if R^c is metrizable, $\widetilde{HD}$-compatible, and lies below R^*. Note that R^c is then Green-compatible. An important example:

PROPOSITION. *The Kuramochi compactification R^k of R is of type G.*

In fact, metrizability and $\widetilde{HD}$-compatibility of R^k are immediate consequences of related results of Constantinescu-Cornea [2, pp. 171 and 169]. That R^k lies below R^* follows from the definition of the Kuramochi compactification given in [2, p. 167].

R^k is actually the only significant compactification of type G known thus far. For a general discussion of its properties we also refer to [5].

3. Distinguished bundles.

8. Let R^c be a compactification of R of type G. We call a bundle $\mathfrak{A}\subset\mathfrak{B}$ *R^c-distinguished* if $m(\mathfrak{A})>0$ and there exists a point $p\in\Gamma^c$ such that $e^c(l)=p$ for almost every $l\in\mathfrak{A}$. The point p will be referred to as the *end* of $\mathfrak{A}$. In the case $R^c=R^k$ we simply say that $\mathfrak{A}$ is *distinguished*.

We shall characterize points $p\in\Gamma^c$ of positive measure in terms of R^c-distinguished bundles:

THEOREM. *Let R^c be a compactification of type G of a hyperbolic Riemann surface R. A point $p\in\Gamma^c=R^c-R$ has positive harmonic measure if and only if there exists an R^c-distinguished bundle $\mathfrak{A}$ with end p.*

The proof will be given in 9–13.

9. Let $\Gamma=R^*-R$ be the Royden boundary of R. For $l\in\mathfrak{B}_r$ denote by $e(l)$ the set $\bar{l}-l\cup\{z_0\}$ in Γ, with $\bar{l}$ the closure of l in R^*. Given a subset $S\subset\Gamma$ we write

$$(8)\qquad \tilde{S}=\{l\in\mathfrak{B}\,|\,e(l)\cap S\neq\emptyset\},\qquad \check{S}=\{l\in\mathfrak{B}\,|\,e(l)\subset S\}\ .$$

We shall employ the following auxiliary result ([7], [8]): For every F_σ-set K (resp. G_δ-set U) in Γ

$$(9)\qquad \bar{m}(\tilde{K})\leqq\mu(K),\qquad \underline{m}(\check{U})\geqq\mu(U)\ ,$$

where $\bar{m}$ and $\underline{m}$ are the outer and inner measures induced by m.

Let p^* be on the Royden *harmonic boundary* Δ of R. The set

$$\Lambda_{p^*}=\{q^*\in\Gamma\,|\,u(q^*)=u(p^*)\ \text{for all}\ u\in HBD(R)\}$$

is called a *block* at p^*. It is known ([7], [8]) that it has a measurable $\tilde{\Lambda}_{p^*}$,

$$(10)\qquad m(\tilde{\Lambda}_{p^*})=\mu(p^*)\ ,$$

and that

$$(11)\qquad u(p^*)=\lim_{z\in l,\,r(z)\to 1}u(z)$$

for every $u\in HD(R)$ and almost every $l\in\tilde{\Lambda}_{p^*}$.

10. Suppose $\mathfrak{A}$ is an R^c-distinguished bundle with end $p \in \Gamma^c$. We are to prove that $\mu^c(p) > 0$. Take the projection $\pi = \pi^c$ of R^* onto R^c (see 5). The set $K = \pi^{-1}(p)$ is compact and clearly $\mathfrak{A} \subset \tilde{K}$. By (9),

$$0 < m(\mathfrak{A}) \leqq \bar{m}(\tilde{K}) \leqq \mu(K) .$$

From (6) it follows that $\mu(K) = \mu(\pi^{-1}(p)) = \pi^c(p)$. Therefore

$$0 < m(\mathfrak{A}) \leqq \mu^c(p) .$$

11. Conversely suppose that $p \in \Gamma^c$ and $\mu^c(p) > 0$. Since R^c is $\widetilde{HD}$-compatible, the function $u(z) = \int_p P^c(z, q) d\mu^c(q)$ is $\widetilde{HD}$-minimal on R. By (6) we see that

$$u(z) = \int_{\pi^{-1}(p)} P(z, q^*) d\mu(q^*) . \tag{12}$$

Since R^* is also $\widetilde{HD}$-compatible and the integral representation (12) of the $\widetilde{HD}$-function u is unique up to a boundary function vanishing μ-almost everywhere on Γ ([6], [8]), we conclude that there exists a point $p^* \in \pi^{-1}(p)$ with $\mu(p^*) = \mu(\pi^{-1}(p)) > 0$. Observe that

$$m(\tilde{\Lambda}_{p^*}) = \mu(p^*) > 0 . \tag{13}$$

In view of the Green-compatibility of R^c, there exists a measurable subset $\mathfrak{A} \subset \tilde{\Lambda}_{p^*}$ with $m(\tilde{\Lambda}_{p^*}) = m(\mathfrak{A})$ and such that $e^c(l)$ is a single point in Γ^c for each $l \in \mathfrak{A}$.

To conclude that $\mathfrak{A}$ is an R^c-distinguished bundle with end p, we must show that $\mathfrak{A}' = \{l \in \mathfrak{A} \mid e^c(l) \neq p\}$ is of m-measure zero. For this purpose take a sequence $\{U_n\}_1^\infty$ of open sets in Γ^c with

$$U_{n+1} \subset \bar{U}_{n+1} \subset U_n, \qquad \bigcap_1^\infty U_n = \{p\} .$$

Let $\mathfrak{A}'_n = \{l \in \mathfrak{A}' \mid e^c(l) \notin U_n\}$. Since $\mathfrak{A}' = \bigcup_{n=1}^\infty \mathfrak{A}'_n$, it suffices to show that $m(\mathfrak{A}'_n) = 0$ for every n.

12. First we assume that $R \notin O_{HD}$. For an arbitrarily fixed n there exists a $u_n \in HBD(R)$ such that

$$0 \leqq u_n | \Delta \leqq 1,\ u_n | \pi^{-1}(U_{n+1}) \cap \Delta = 1,\ u_n | (\Delta - \pi^{-1}(U_n)) = 0 . \tag{14}$$

In view of (11), there exists a measurable subset $\mathfrak{A}''_n \subset \mathfrak{A}'_n$ with $m(\mathfrak{A}'_n - \mathfrak{A}''_n) = 0$ and

$$1 = u_n(p^*) = \lim_{z \in l, r(z) \to 1} u_n(z) \tag{15}$$

for every $l \in \mathfrak{A}''_n$. The set $E_n = \{q^* \in \Gamma \mid u_n(q^*) < \frac{1}{2}\}$ is open in Γ. By

(15), $e(l)\cap E_n=\varnothing$ for every $l\in\mathfrak{A}_n''$. Because of the definition of $\mathfrak{A}_n'$, it is also clear that $e(l)\cap\pi^{-1}(U_n)=\varnothing$ for every $l\in\mathfrak{A}_n''$. Since the set $K_n=\Gamma-\pi^{-1}(U_n)\cup E_n$ is compact and $\pi^{-1}(U_n)\cup E_n\supset\Delta$, we have $K_n\subset\Gamma-\Delta$ and a fortiori $\mu(K_n)=0$.

On the other hand, $e(l)\subset K_n$ for every $l\in\mathfrak{A}_n''$. Therefore $\mathfrak{A}_n''\subset\check{K}_n\subset\tilde{K}_n$. In view of (9), we obtain

$$m(\mathfrak{A}_n'')\leqq\bar{m}(\tilde{K}_n)\leqq\mu(K_n)=0$$

and conclude that $m(\mathfrak{A}_n')=m(\mathfrak{A}_n'')=0$.

13. If $R\in O_{HD}$, then Δ consists of a single point and consequently $\Delta=\{p^*\}$. The set $F_n=\Gamma-\pi^{-1}(U_n)$ is compact in $\Gamma-\Delta$ and hence $\mu(F_n)=0$. By the definition of $\mathfrak{A}_n'$ we have $\mathfrak{A}_n'\subset\check{F}_n\subset\tilde{F}_n$. Therefore $m(\mathfrak{A}_n')\leqq m(\tilde{F}_n)\leqq\mu(F_n)=0$. The proof of Theorem 8 is herewith complete.

4. Characterization of distinguished bundles.

14. We next give necessary and sufficient conditions for a bundle to be distinguished, without referring to its end:

THEOREM. *Let R^c be a compactification of type G of a hyperbolic Riemann surface R. A bundle $\mathfrak{A}\subset\mathfrak{B}$ is R^c-distinguished if and only if $m(\mathfrak{A})>0$ and for each $u\in HD(R)$ there exists a number c_u such that*

$$\lim_{z\in l,\,r(z)\to1}u(z)=c_u \tag{16}$$

for almost every $l\in\mathfrak{A}$.

The proof will be given in 15–18.

15. First suppose $\mathfrak{A}$ is R^c-distinguished with end $p\in\Gamma^c$. Then by 10 and 11, there exists a point $p^*\in K=\pi^{-1}(p)$ such that

$$0<\mu^c(p)=\mu(K)=\mu(p^*)\,.$$

Fix a $u\in HD(R)$. By the Godefroid theorem [3] (see also [7], [8]),

$$u(l)=\lim_{z\in l,\,r(z)\to1}u(z) \tag{17}$$

exists for almost every $l\in\mathfrak{B}_r$. On omiting from $\mathfrak{A}$ a set of measure zero we may assume that $u(l)$ in (17) exists for every $l\in\mathfrak{A}$. We may also suppose that $e^c(l)=p$ and a fortiori $e(l)\subset K$ for every $l\in\mathfrak{A}$.

Since $\mu(p^*)>0$, $|u(p^*)|<\infty$ (cf. [6], [8]). Let

$$\mathfrak{A}'=\{l\in\mathfrak{A}\,|\,u(l)-u(p^*)\neq0\}$$

and

$$K_n = \{q^* \in K \,|\, |u(q^*) - u(p^*)| \geqq 1/n\} .$$

Clearly K_n is a compact set. For $l \in \mathfrak{A}'$ and $q^* \in e(l)$, we have $u(l) = u(q^*)$ by (17) and the continuity of u on R^*. Therefore $|u(q^*) - u(p^*)| \geqq 1/n$ for some n and a fortiori $e(l) \subset K_n$. It follows that

$$\mathfrak{A}' \subset \bigcup_{n=1}^{\infty} \check{K}_n \subset \bigcup_{n=1}^{\infty} \tilde{K}_n ,$$

which by (9) gives

$$m(\mathfrak{A}') \leqq \bar{m}\Big(\bigcup_{n=1}^{\infty} \tilde{K}_n\Big) \leqq \sum_{n=1}^{\infty} \bar{m}(\tilde{K}_n) \leqq \sum_{n=1}^{\infty} \mu(K_n) .$$

From $K_n \subset K - p^*$ and $\mu(K) = \mu(p^*)$, we obtain $\mu(K_n) = 0$. Consequently $m(\mathfrak{A}') = 0$ and, since

$$\lim_{z \in l, r(z) \to 1} u(z) = u(l) = u(p^*)$$

for every $l \in \mathfrak{A} - \mathfrak{A}'$, we have (16) for almost every $l \in \mathfrak{A}$.

16. Conversely suppose that, for a bundle $\mathfrak{A} \subset \mathfrak{B}$ with $m(\mathfrak{A}) > 0$, (16) is satisfied. We may assume that $e^c(l)$ is a single point in Γ^c for every $l \in \mathfrak{A}$.

First consider the case $R \in O_{HD}$. The harmonic boundary Δ consists of a single point p^* and $\mu(p^*) > 0$. Let $p = \pi(p^*)$. Take a sequence $\{U_n\}_1^\infty$ of open sets in Γ^c such that $\bar{U}_{n+1} \subset U_n$ and $\bigcap_1^\infty U_n = \{p\}$. For the bundles $\mathfrak{A}'_n = \{l \in \mathfrak{A} \,|\, e^c(l) \notin U_n\}$, $n = 1, 2, \cdots$, and

$$\mathfrak{A}' = \{l \in \mathfrak{A} \,|\, e^c(l) \neq p\}$$

we have $\mathfrak{A}' = \bigcup_1^\infty \mathfrak{A}'_n$. Set $K_n = \Gamma - \pi^{-1}(U_n) \subset \Gamma - \Delta$. Every $l \in \mathfrak{A}'_n$ has $e(l) \subset K_n$ and we obtain $\mathfrak{A}'_n \subset \check{K}_n \subset \tilde{K}_n$. Hence

$$m(\mathfrak{A}'_n) \leqq \bar{m}(\tilde{K}_n) \leqq \mu(K_n) = 0$$

and therefore $m(\mathfrak{A}') = 0$, i.e., $e^c(l) = p$ for almost every $l \in \mathfrak{A}$. This proves that $\mathfrak{A}$ is R^c-distinguished.

17. Next suppose $R \notin O_{HD}$. The family

$$T(\mathfrak{A}) = \{u \in HBD(R) \,|\, 0 \leqq u \leqq 1 \text{ on } R,\ u(l) = 1 \text{ for almost every } l \in \mathfrak{A}\}$$

is a Perron family and

$$s(z) = \inf\{u(z) \,|\, u \in T(\mathfrak{A})\} \tag{18}$$

is an $\widetilde{HD}$-minimal function on R (see [7], [8]). We can therefore choose a decreasing sequence $\{h_n\} \subset T(\mathfrak{A})$ such that

(19) $$s(z) = \lim_n h_n(z)$$

on R. Let $\mathfrak{A}_0$ be a measurables subset of $\mathfrak{A}$ with $m(\mathfrak{A}) = m(\mathfrak{A}_0)$ such that $h_n(l)$ exists and equals unity for every $n = 1, 2, \cdots$, and every $l \in \mathfrak{A}_0$. We set

$$\bar{s}(l) = \limsup_{z \in l, r(z) \to 1} s(z)$$

and observe that

$$s(z_0) = \int_{\mathfrak{B}} s(re^{il})dm(l) \leqq \int_{\mathfrak{B}} h_n(re^{il})dm(l) = h_n(z_0)$$

for every $r \in (0, 1)$ (see [7], [8]). By Fatou's lemma

$$s(z_0) \leqq \int_{\mathfrak{B}} \bar{s}(l)dm(l) \leqq \int_{\mathfrak{B}} h_n(l)dm(l) = h_n(z_0) .$$

Let $h(l) = \lim_n h_n(l)$. Since $h_n(l) \geqq \bar{s}(l)$ and

$$0 \leqq \int_{\mathfrak{B}} (h(l) - \bar{s}(l))dm(l) \leqq \lim_{n\to\infty} (h_n(z_0) - s(z_0)) = 0 ,$$

we conclude that $\bar{s}(l) = h(l)$ almost everywhere on $\mathfrak{B}$. In view of $h(l) = 1$ for every $l \in \mathfrak{A}_0$ we may suppose that

(20) $$\bar{s}(l) = 1 \qquad (l \in \mathfrak{A}) .$$

18. The remainder of the proof is analogous to that in 11–12. In fact, since s is $\widetilde{HD}$-minimal, there exist points p and p^* in Γ^c and Γ respectively such that $\mu^c(p) = \mu(p^*) > 0$, $p^* \in \pi^{-1}(p)$, and

$$s(z) = \int_p P^c(z, q)d\mu^c(q) = \int_{p^*} P(z, q^*)d\mu(q^*) .$$

We wish to show that $e^c(l) = p$ for almost every $l \in \mathfrak{A}$, that is, $\mathfrak{A}$ is R^c-distinguished with end p. For this purpose set $\mathfrak{A}' = \{l \in \mathfrak{A} \mid e^c(l) \neq p\}$. To see that $m(\mathfrak{A}') = 0$ take a sequence $\{U_n\}$ of open sets in Γ^c such that

$$\bar{U}_{n+1} \subset U_n, \qquad \bigcap_1^\infty U_n = \{p\} .$$

For $\mathfrak{A}'_n = \{l \in \mathfrak{A} \mid e^c(l) \notin U_n\}$ we have $\mathfrak{A}' = \bigcup_1^\infty \mathfrak{A}'_n$ and it suffices to show that $m(\mathfrak{A}'_n) = 0$ for every $n = 1, 2, \cdots$. Take a function $u_n \in HBD(R)$ with

$$0 \leqq u_n | \Delta \leqq 1,\ u_n | \pi^{-1}(U_{n+1}) \cap \Delta = 1,\ u_n | (\Delta - \pi^{-1}(U_n)) = 0 .$$

We may suppose $u_n(l)$ exists for every $l \in \mathfrak{A}$. Since $1 \geqq u_n \geqq s$ on R, (20) implies that

(21) $$u_n(l) = 1 \qquad (l \in \mathfrak{A}) .$$

Clearly $e(l) \subset \Gamma - \pi^{-1}(U_n)$ for every $l \in \mathfrak{A}'_n$. Moreover, if we set $E_n = \{q^* \in \Gamma \mid u_n(q^*) < \frac{1}{2}\}$, then $e(l) \subset \Gamma - E_n \cup \pi^{-1}(U_n) = K_n$ for every $l \in \mathfrak{A}'_n$. Since K_n is compact and contained in $\Gamma - \Delta$,

$$\mathfrak{A}'_n \subset \check{K}_n \subset \tilde{K}_n$$

implies that

$$m(\mathfrak{A}'_n) \leqq \bar{m}(\tilde{K}_n) = \mu(K_n) = 0 .$$

The proof of Theorem 14 is herewith complete.

5. Conclusion.

19. Recall that a bundle $\mathfrak{A} \subset \mathfrak{B}$ is distinguished with end p on the Kuramochi boundary if $m(\mathfrak{A}) > 0$ and almost every Green line in $\mathfrak{A}$ terminates at p. Since the Kuramochi compactification is of type G, Theorems 8 and 14 imply:

THEOREM. *A point p of the Kuramochi boundary of a hyperbolic Riemann surface R has positive measure if and only if there exists a distinguished bundle $\mathfrak{A}$ of Green lines with end p.*

A bundle $\mathfrak{A}$ of Green lines with $m(\mathfrak{A}) > 0$ is distinguished if and only if, for every $u \in HD(R)$, there exists a number c_u such that the "radial limit" $\lim_{z \in l, r(z) \to 1} u(z)$ *exists and equals c_u for almot every $l \in \mathfrak{A}$.*

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