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ON GENERALIZED TRANSLATED QUASI-CESÀRO SUMMABILITY

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Let $\alpha > 0, \beta > -1$. The (C_t, α, β) transformation of the sequence $\{s_k\}$ is defined by

$$t_n = \frac{\Gamma(\beta+n+2)\Gamma(\alpha+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(\alpha+\beta+n+k+2)}s_k ,$$

and the (C_t, α, β) transformation of the function $s(x)$ is defined by

$$g(y) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}s(x)}{(x+y)^{\alpha+\beta+1}}dx .$$

Some properties of the above two transformations are given in this paper and the relation between the summability methods defined by these transformations is discussed.

1. For any sequence $\{\mu_n\}$ the Hausdorff summability (H, μ_n) is defined by the transformation

$$t_n = \sum_{k=0}^{n}\binom{n}{k}(\Delta^{n-k}\mu_k)s_k ,$$

where

$$\begin{aligned}\Delta^0_{\mu_k} &= \mu_k ,\\ \Delta\mu_k &= \mu_k - \mu_{k+1} ,\\ \Delta^m_{\mu_k} &= \Delta\Delta^{m-1}\mu_k .\end{aligned}$$

Transposing the matrix of the (H, μ_n), transformation we get the matrix of the quasi-Hausdorff transformation

$$t_n = \sum_{k=n}^{\infty}\binom{k}{n}(\Delta^{k-n}\mu_n)s_k ,$$

which will be denoted by (H^*, μ_n). Ramanujan [8] introduced the (S, μ_n) summability, which is defined by the transformation

$$t_n = \sum_{k=0}^{\infty}\binom{k+n}{n}(\Delta^k\mu_n)s_k .$$

Thus the elements of row n of the matrix of the (S, μ_n) transformation are those of the corresponding row of the (H^*, μ_n) transformation moved n places to the left.

It is known [8] that if (H, μ_n) is regular and if $\mu_n \to 0$ as $n \to \infty$, then (S, μ_{n+1}) is regular; conversely, if (S, μ_{n+1}) is regular, then (H, μ_n)

can be made regular by a suitable choice of μ_0.

When

$$\mu_n = \frac{1}{\binom{n+\alpha}{n}},$$

(H, μ_n) reduces to the Cesàro summability (C, α). Borwein [3] introduced the generalized Cesàro summability (C, α, β) which is (H, μ_n) with

$$\mu_n = \frac{\binom{n+\beta}{n}}{\binom{n+\alpha+\beta}{n}}. \tag{1}$$

The aim of this paper is to discuss properties of the (S, μ_{n+1}) summability with μ_n given by (1) for $\alpha > 0, \beta > -1$ and of the analogous functional transformation. We shall denote this summability by (C_t, α, β). The case in which $\beta = 0$ has been considered by Kuttner [6] and a summability method similar to (C_t, α, β) has been discussed by me [7].

A straightforward calculation shows that the (C_t, α, β) transformation is given by

$$\begin{aligned} t_n = t(n, \alpha, \beta) &= \frac{(\beta+1)(\beta+2)\cdots(\beta+n+1)}{n!} \\ &\quad \times \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)(k+1)(k+2)\cdots(k+n)}{(\alpha+\beta+1)(\alpha+\beta+2)\cdots(\alpha+\beta+n+1+k)} s_k \\ &= \frac{\Gamma(\beta+n+2)\Gamma(\alpha+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(\alpha+\beta+n+k+2)} s_k . \end{aligned} \tag{2}$$

It is clear that, if (2) converges for one value of n, then it converges for all n. Further, a necessary and sufficient condition for this to happen is that

$$\sum_{k=1}^{\infty} \frac{s_k}{k^{\beta+2}} \tag{3}$$

should converge.

Let $s(x)$ be any function L-integrable in any finite interval of $x \geqq 0$ and bounded in some right-hand neighbourhood of the origin. Let $\alpha > 0, \beta > -1$, and let

$$g(y) = g(y, \alpha, \beta) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1} s(x)}{(x+y)^{\alpha+\beta+1}} dx . \tag{4}$$

If $g(y)$ exists for $y > 0$ and if

$$\lim_{y\to\infty} g(y) = s ,$$

we say that $s(x)$ is summable (C_t, α, β) to s.

It is clear that a necessary and sufficient condition for the convergence of (4) is that

$$\int_1^\infty \frac{s(x)}{x^{\beta+2}}dx \tag{5}$$

should converge.

2. The relationship between sequence-to-sequence and function-to-functions transformations. Given any sequence $\{s_n\}$, let the function $f(x)$ be defined by

$$f(x) = s_n \quad (n \leqq x < n + 1;\, n = 0, 1, 2, \cdots) .$$

Then the (C_t, α, β) summability of $\{s_n\}$ is equivalent to the (C_t, α, β) summability of $f(x)$ for $\alpha > 0$, $\beta = 0$ (see [6] Theorem 4). However, the proof breaks down when $\beta > 0$. We can prove that they are equivalent for $-1 < \beta \leqq 0$ as follows. Write

$$a(n, k) = \frac{\Gamma(\alpha + k)\Gamma(k + n + 1)}{\Gamma(k + 1)\Gamma(\alpha + \beta + n + k + 2)}$$

$$b(y, k) = \int_k^{k+1} \frac{x^{\alpha-1}}{(x + y)^{\alpha+\beta+1}}dx .$$

As in [6], we may suppose that $s_0 = 0$. Then the result would follow if, corresponding to equation (11) of [6], we proved that, if (3) converges, then uniformly for $0 \leqq \theta < 1$,

$$\sum_{k=1}^\infty [a(n, k) - b(n + \theta, k)]s_k = o\left(\frac{1}{n^{\beta+1}}\right) . \tag{6}$$

Choose an integer Q such that $Q \geqq \beta + 3$. From equations analogous to those of the last line and line 6 from bottom of p. 709 of [6], we find that

$$a(n, k) - b(n + \theta, k) = \Sigma p(\theta)\frac{k^{\alpha-q}}{(n + k)^{\alpha+\beta+r}} + O\left(\frac{k^{\alpha-Q}}{(k + n)^{\alpha+\beta+1}}\right) , \tag{7}$$

where $p(\theta)$ is a polynomial in θ (which may be different for each term in the sum), and the sum is taken over those integers q, r which are such that

$$q \geqq 1,\, r \geqq 1, \quad q, r \text{ not both } 1, \quad q + r \leqq Q .$$

Since the convergence of (3) implies that

$$s_k = o(k^{\beta+2}) ,$$

and since $\alpha > 0$, $Q \geqq \beta + 3$, we see that the contribution to the expression on the left of (6) of the "0" term in (7) is

$$o\left(\frac{1}{n^{\beta+1}}\right).$$

Hence the result would follow if (corresponding to Lemma 2 of [6]) we could prove that the convergence of (3) implied that, for relevant q, r,

$$\sum_{k=1}^{\infty} \frac{k^{\alpha-q}}{(k+n)^{\alpha+\beta+r}} s_k = o\left(\frac{1}{n^{\beta+1}}\right). \tag{8}$$

Now write

$$v_k = \sum_{m=k}^{\infty} \frac{s_m}{m^{\beta+2}}$$

so that $v_k \to 0$ (and this is all we know). The sum on the left of (8) is

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{k^{\alpha+\beta+2-q}}{(k+n)^{\alpha+\beta+r}}(v_k - v_{k+1}) \\ &= \frac{v_1}{(n+1)^{\alpha+\beta+r}} + \sum_{k=2}^{\infty} v_k \left\{ \frac{k^{\alpha+\beta+2-q}}{(k+n)^{\alpha+\beta+r}} - \frac{(k-1)^{\alpha+\beta+2-q}}{(k-1+n)^{\alpha+\beta+r}} \right\}. \end{aligned} \tag{9}$$

The first term on the right of (9) is $o(1/n^{\beta+1})$ (since $r \geqq 1, \alpha > 0$). The expression in curly brackets in the second term is

$$O\left(\frac{k^{\alpha+\beta+1-q}}{(k+n)^{\alpha+\beta+r}}\right)$$

(and this result is best possible). This gives the required result when $\beta \leqq 0$; but if $\beta > 0$, all that we can deduce in the "worst" cases (which are $q = 1$, $r = 2$ or $q = 2$, $r = 1$) is that the sum (9) is $o(1/n)$.

Of course, the fact that the proof breaks down does not imply that the theorem itself is false. My guess is that the theorem probably is false for $p > 0$; but I have not actually got a counter example.

3. Theorems. The following two theorems with $\beta = 0$ are Theorem 1′ and Theorem 2′ given by Kuttner [6]. The proof of Theorem 1 is similar to that of Theorem 1′ in [6], and Theorem 2 follows from Lemma 1 and Lemma 2 of this paper.

THEOREM 1. *Let $\alpha > 0, \beta > -1$ and $r \geqq 0$ and let $s(x)$ be summable (C, r)*[1] *to s and (4) converge. Then $s(x)$ is summable (C_t, α, β) to s.*

THEOREM 2. *Let $\alpha > \alpha' > 0, \beta > -1$, and let $s(x)$ be summable (C_t, α, β) to s. Then $s(x)$ is summable (C_t, α', β) to s.*

[1] For definition of the (C, r) summability of $s(v)$, see [7].

In § 5, we shall prove

THEOREM 3. *Let $\alpha > 0$, $\beta > \beta' > -1$. Suppose that $s(x)$ is summable (C_t, α, β) to s and the integral*

$$\int_1^\infty \frac{s(x)}{x^{\beta'+2}} dx$$

converges. Then $s(x)$ is summable (C_t, α, β') to s.

The sequence $\{s_n\}$ is said to be summable A_λ to s if

$$f_\lambda(x) = (1-x)^{\lambda+1} \sum_{n=0}^\infty \binom{n+\lambda}{n} s_n x^n$$

converges for all x in the interval $0 \leqq x < 1$ and tend to a finite limit s as $x \to 1-$. The A_0 method is the ordinary Abel method.

It is known (see [1] and [2]) that $A_\mu \supset A_\lambda$ for $\lambda > \mu > -1$. For other properties of this summability method, see [1] and [6]. We shall prove

THEOREM 4. *Let $\lambda > -1$, $\beta > -1$. Suppose that the sequence $\{s_n\}$ is summable A_λ to s and that* (3) *converges. Then the sequence is summable $(C_t, \lambda + 1, \beta)$ to s.*

4. Lemmas.

LEMMA 1. *Let $\alpha > \alpha' > 0$, $\beta > -1$. Suppose that* (5) *converges. Then*

$$y^{\alpha-1} g(y, \alpha', \beta) = \frac{\Gamma(\alpha)}{\Gamma(\alpha')\Gamma(\alpha-\alpha')} \int_0^y t^{\alpha'-1}(y-t)^{\alpha-\alpha'-1} g(t, \alpha, \beta) dt .$$

The proof of this lemma is similar to that of Lemma 4 in [6].

LEMMA 2. *Let*

$$t(x) = \int_0^\infty c(x, y) s(y) dy .$$

Then in order that

$$s(y) \to s \qquad (y \to \infty)$$

should imply

$$t(x) \to s \qquad (x \to \infty)$$

for every bounded $s(y)$, it is sufficient that

$$\int_0^\infty |c(x, y)| dy < H ,$$

where H is independent of x, that

$$\int_0^Y |c(x, y)|dy \to 0$$

when $x \to \infty$, *for every finite* Y, *and that*

$$\int_0^\infty c(x, y)dy \to 1$$

when $x \to \infty$.

This Theorem 6 in [4].

5. Proof of Theorem 3. Let

$$\phi(x) = \int_x^\infty \frac{s(u)}{u^{\beta+2}} du$$

for $x > 0$. Then $\phi(x)$ is continuous in $(0, \infty)$, and $\phi(x) \to 0$ as $x \to \infty$; hence $\phi(x)$ is bounded in (B, ∞) for any $B > 0$, say

$$|\phi(x)| \leqq M$$

for $x \geqq B$, where M may depend on B if B is small, but may be taken as an absolute constant for large B. It follows that

$$\begin{aligned}\left|\int_B^\infty \frac{x^{\alpha-1}s(x)}{(x+t)^{\alpha+\beta+1}}dx\right| &= \left|\int_B^\infty \left(\frac{x}{x+t}\right)^{\alpha+\beta+1} d\phi(x)\right| \\ &= \left|\left(\frac{B}{B+t}\right)^{\alpha+\beta+1}\phi(B)\right. \\ &\qquad \left. + (\alpha+\beta+1)t\int_B^\infty \frac{1}{(x+t)^2}\left(\frac{x}{x+t}\right)^{\alpha+\beta}\phi(x)dx\right| \\ &\leqq |\phi(B)| + (\alpha+\beta+1)tM\int_B^\infty \frac{dx}{(x+t)^2} \\ &\leqq (\alpha+\beta+2)M\,.\end{aligned} \tag{10}$$

Since $s(x)$ is bounded in some right-hand neighbourhood of the origin, there exists $B_0 > 0$ such that

$$|s(x)| \leqq K$$

for $0 < x < B_0$. By partial integration, we obtain

$$\left|t^{\beta+1}\int_0^{B_0} \frac{x^{\alpha-1}s(x)}{(x+t)^{\alpha+\beta+1}}dx\right| \leqq \frac{K(\alpha+2\beta+2)}{\alpha(\beta+1)}\,. \tag{11}$$

By combining (10) and (11) it follows that $g(t, \alpha, \beta)$ is bounded in any finite interval $(0, T)$. Since it tends to s as $t \to \infty$, $g(t, \alpha, \beta)$ is bounded in $(0, \infty)$. Thus, for $y > 0$, the integral

$$I = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \int_y^\infty t^{-\beta-1}(t-y)^{\beta-\beta'-1} g(t, \alpha, \beta) dt$$

converges. In view of the definition of $g(t, \alpha, \beta)$ it follows that

$$I = \lim_{A \to \infty} I(A) \tag{12}$$

where

$$I(A) = \int_y^A (t-y)^{\beta-\beta'-1} dt \int_0^\infty \frac{x^{\alpha-1}s(x)}{(x+t)^{\alpha+\beta+1}} dx .$$

It follows from (10) by dominated convergence that, for fixed A,

$$\int_y^A (t-y)^{\beta-\beta'-1} dt \int_B^\infty \frac{x^{\alpha-1}s(x)}{(x+t)^{\alpha+\beta+1}} dx \to 0$$

as $B \to \infty$. Hence, by Fubini's theorem

$$I(A) = \int_0^\infty x^{\alpha-1}s(x) dx \int_y^A \frac{(t-y)^{\beta-\beta'-1}}{(x+t)^{\alpha+\beta+1}} dt . \tag{13}$$

We will now show that, for fixed y,

$$\int_0^\infty x^{\alpha-1}s(x) dx \int_A^\infty \frac{(t-y)^{\beta-\beta'-1}}{(x+t)^{\alpha+\beta+1}} dt \to 0 \tag{14}$$

as $A \to \infty$. It is clear that for large A the inner integral in (14) is $O(A^{-\alpha-\beta'-1})$ uniformly in $0 \leqq x \leqq 1$, so that the contribution to (14) of the range $0 < x < 1$ tends to 0 as $A \to \infty$. Now write

$$\psi(x) = \int_x^\infty \frac{s(u)}{u^{\beta'+2}} du;$$

thus we are given that $\psi(x)$ exists and that it tends to 0 as $x \to \infty$. The contribution to (14) of $x > 1$ may now be written

$$-\int_1^\infty x^{\alpha+\beta'+1} d\psi(x) \int_A^\infty \frac{(t-y)^{\beta-\beta'-1}}{(x+t)^{\alpha+\beta+1}} dt . \tag{15}$$

It is easily seen that, for fixed y, A and large x, the inner integral in (15) is $O(x^{-\alpha-\beta'-1})$; thus, integrating by parts, (15) becomes

$$\begin{aligned} &\psi(1) \int_A^\infty \frac{(t-y)^{\beta-\beta'-1}}{(1+t)^{\alpha+\beta+1}} dt \\ &+ \int_1^\infty x^{\alpha+\beta'} \psi(x) dx \int_A^\infty \frac{(t-y)^{\beta-\beta'-1}[(\alpha+\beta'+1)t - (\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}} dt . \end{aligned} \tag{16}$$

Now for fixed y and large A, uniformly in $0 \leqq x \leqq A$, the inner integral in (16) is

$$O\left\{\int_A^\infty t^{-\alpha-\beta'-2}dt\right\} = O(A^{-\alpha-\beta'-1}) .$$

Hence

$$\int_1^A x^{\alpha+\beta'}\psi(x)dx\int_A^\infty \frac{(t-y)^{\beta-\beta'-1}[(\alpha+\beta'+1)t-(\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}}dt$$
$$= \left(\int_1^{A/\log A} + \int_{A/\log A}^A\right)x^{\alpha+\beta'}\psi(x)O(A^{-\alpha-\beta'-1})dx$$
$$= O\left(A^{-\alpha-\beta'-1}\int_1^{A/\log A} x^{\alpha+\beta'}dx\right)$$
$$+ O\left(A^{-\alpha-\beta'-1}\sup_{x\geqq(A/\log A)}|\psi(x)|\int_{A/\log A}^A x^{\alpha+\beta'}dx\right) = O(1) .$$

Nothing that for fixed y and large t

$$(t-y)^{\beta-\beta'-1} = t^{\beta-\beta'-1} + O(t^{\beta-\beta'-2}) ,$$

and also that

$$\int_0^\infty \frac{t^{\beta-\beta'-1}[(\alpha+\beta'+1)t-(\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}}dt = 0 ,$$

we see that, for large A uniformly in $x \geqq A$, the inner integral in (16) is

$$-\int_0^A \frac{t^{\beta-\beta'-1}[(\alpha+\beta'+1)t-(\beta-\beta')x]}{(x+t)^{\alpha+\beta+2}}dt$$
$$+ O\left\{\int_A^\infty \frac{t^{\beta-\beta'-2}|(\alpha+\beta'+1)t-(\beta-\beta')x|}{(x+t)^{\alpha+\beta+2}}dt\right\}$$
$$= O\left\{x^{-\alpha-\beta-1}\int_0^A t^{\beta-\beta'-1}dt\right\} + O\left\{x^{-\alpha-\beta-1}\int_A^x t^{\beta-\beta'-2}dt\right\} + O\left\{\int_x^\infty t^{-\alpha-\beta'-3}dt\right\}$$
$$= O(x^{-\alpha-\beta-1}A^{\beta-\beta'}) + O(x^{-\alpha-\beta'-2})$$

(except that, in the case $\beta-\beta'=1$, we must insert an extra term $O(x^{-\alpha-\beta-1}\log x)$). It is now clear that the expression (16) tends to 0 as $A\to\infty$, and this completes the proof of (14). We deduce from (12), (13) and (14) that

$$I = \int_0^\infty x^{\alpha-1}s(x)dx\int_y^\infty \frac{(t-y)^{\beta-\beta'-1}}{(x+t)^{\alpha+\beta+1}}dt$$
$$= \frac{\Gamma(\beta-\beta')\Gamma(\alpha+\beta'+1)}{\Gamma(\alpha+\beta+1)}\int_0^\infty \frac{x^{\alpha-1}s(x)}{(x+y)^{\alpha+\beta'+1}}dx$$
$$= \frac{\Gamma(\beta-\beta')\Gamma(\alpha)\Gamma(\beta'+1)}{\Gamma(\alpha+\beta+1)}y^{-\beta'-1}g(y,\alpha,\beta') .$$

Thus, in view of the definition of I, we have

$$g(y, \alpha, \beta') = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \beta')\Gamma(\beta' + 1)} y^{\beta'+1}\int_y^\infty t^{-\beta-1}(t - y)^{\beta-\beta'-1}g(t, \alpha, \beta)dt .$$

The kernel of this last transformation can easily be verified to satisfy the conditions of Lemma 2, and the theorem now follows.

6. **Proof of Theorem 4.** It follows from the convergence of (3) that for $\beta > -1$, $s_\nu = o(\nu^{\beta+2})$. We can easily prove that the function $t^{n+k}(1 - t)^{\lambda+\beta'+1}$ has a maximum when

$$t = \frac{k + n}{k + n + \lambda + \beta' + 1} .$$

For large $k + n$, this maximum is $O((k + n)^{-\lambda-\beta'-1})$. Hence, if $\beta' > \beta + 2$, we have, the inversion in the order of integration and summation being justified by absolute convergence,

$$\begin{aligned} &\frac{\Gamma(\beta' + n + 2)}{\Gamma(n + 1)\Gamma(\beta' + 1)}\int_0^1 t^n(1 - t)^{\lambda+\beta'+1}\left\{\sum_{k=0}^\infty \binom{\lambda + k}{k} s_k t^k\right\}dt \\ (17)\quad &= \frac{\Gamma(\beta' + n + 2)}{\Gamma(n + 1)\Gamma(\beta' + 1)}\sum_{k=0}^\infty \binom{\lambda + k}{k} s_k \int_0^1 t^{k+n}(1 - t)^{\lambda+\beta'+1}dt \\ &= \frac{\Gamma(\beta'+n+2)\Gamma(\lambda+\beta'+2)}{\Gamma(n+1)\Gamma(\beta'+1)\Gamma(\lambda+1)}\sum_{k=0}^\infty \frac{\Gamma(\lambda+k+1)\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(\lambda+\beta'+n+k+3)} s_k \\ &= t(n, \lambda + 1, \beta') . \end{aligned}$$

By analytic continuation, (17) holds for $\beta' \geqq \beta$. Hence

$$\begin{aligned} t(n, \lambda + 1, \beta) &= \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)}\int_0^1 t^n(1 - t)^\beta f_\lambda(t)dt \\ &= \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)}\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y} f_\lambda(1 - e^{-y})dy . \end{aligned}$$

By Lemma 2 the result with follow if

$$\text{(i)}\qquad \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)}\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y}dy < H$$

where H is independent of n,

$$\text{(ii)}\qquad \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)}\int_0^Y (1 - e^{-y})^n e^{-(\beta+1)y}dy \to 0$$

when $n \to \infty$, for every finite Y, and

$$\text{(iii)}\qquad \frac{\Gamma(\beta + n + 2)}{\Gamma(n + 1)\Gamma(\beta + 1)}\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y}dy \to 1 ,$$

when $n \to \infty$. Since

$$\int_0^\infty (1 - e^{-y})^n e^{-(\beta+1)y} dy = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(\beta+n+2)},$$

(i) and (iii) are satisfied. We have $\Gamma(n+\beta+2) \sim n^{\beta+1}\Gamma(n+1)$, and the integral in (ii) is, by changing the variable,

$$\int_0^{1-e^{-Y}} t^n(1-t)^\beta dt .$$

Hence (ii) is satisfied.

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