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COMPLETIONS OF DEDEKIND PRIME RINGS AS SECOND ENDOMORPHISM RINGS

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The purpose of this paper is to show that if M is a maximal two-sided ideal of a Dedekind prime ring R and P is any maximal right ideal containing M, then the M-adic completion $\bar{R}$ of R can be realized as the second endomorphism ring of $E = E(R/P)$, the R-injective hull of R/P; that is, as end $({}_K E)$ where $K = \text{end}(E_R)$. The ring K turns out to be a complete, local, principal ideal domain.

This paper was motivated by a result of Matlis [6] which says that if P is a prime ideal of a commutative Noetherian ring R, then the P-adic completion of the localization of R at P can be realized as the ring of endomorphisms of $E = E(R/P)$, the R-injective hull of R/P.

Since $\bar{R}$ is a full matrix ring over a complete local domain L [4], we are able to approach the problem by considering first the case that R is a complete local domain, then by means of the Morita theorems we pass to the case $\bar{R} = R$, and finally pass to the general case.

1. Introduction. A prime ring R is called a Dedekind prime ring if it is Noetherian, hereditary, and a maximal order in its classical quotient ring Q (see [3]). A ring R is called local if the nonunits of R form an ideal.

If R is a Dedekind prime ring with a nonzero prime ideal M, then M is a maximal two-sided ideal and $\cap M^n = 0$ (see Robson [7]). Let $\bar{R} = \bar{R}_M$ be the completion of R at M in the sense of Goldie [3]. In this situation combining results of Goldie ([3], Theorem 4.5) and Gwynne and Robson ([4], Theorem 2.3) yields the following theorem.

THEOREM 1.1. *Let R be a Dedekind prime ring with a maximal ideal M. Then* (i) *$\bar{R}$ has a unique maximal two-sided ideal $\bar{M}$, $\bar{M}$ is the Jacobson radical of $\bar{R}$, and $R \cap \bar{M}^p = M^p$.*

(ii) *$\bar{R}$ is a full $k \times k$ matrix ring over a domain L which has a unique maximal ideal N, and $L/N = F$ where F is a division ring. Also $R/M^p \simeq \bar{R}/\bar{M}^p$* (each coset of $\bar{M}^p$ has a representative in R).

(iii) *$\bar{R}$ is a prime principal ideal ring and L is a complete, local, principal ideal domain. The only one-sided ideals of L are the powers of N.*

For the rest of this section let R, M, $\bar{R}$, $\bar{M}$, L, and N be as in Theorem 1.1. Let x be the generator of N; then $N = xL = Lx$ and $N^k = x^k L = Lx^k$.

2. **The Ring L.** This section will be concerned with the construction of the L-injective hull of $(L/N)_L$ and with showing that Theorem 4.4 holds for L.

LEMMA 2.1. *L/N^k can be embedded in L/N^{k+1} as a right L-module via the map h_k: $L/N^k \to L/N^{k+1}$ defined by $h_k([u + N^k]) = [xu + N^{k+1}]$.*

Proof. h_k is clearly additive and right L-linear. Suppose $h_k([u + N^k]) = [0 + N^{k+1}]$. From the definition of h_k it follows that $xu \in N^{k+1}$ so that $xu = x^{k+1}u$, for some u' in L and $u = x^k u' \in N^k$. Hence $[u + N^k] = [0 + N^k]$ and n_k is a monomorphism. A similar argument shows that h_k is well-defined.

The maps $\{h_k\}$ and the right L-modules $\{(L/N^k)_L\}$ give rise to a directed system. Let E_L be the direct limit of this system. Then E_L can be considered as an ascending union of a family of submodules, $\{(S_j)_L\}$, which is totally ordered by inclusion and where each $(S_j)_L$ is isomorphic to $(L/N^i)_L$.

LEMMA 2.2. *Consider $(L/N^{p+t+1})_L$. Take $a \in N^p/N^{p+t+1}$ and $d \in N^p \backslash N^{p+1}$. The equation $yd = a$ has a solution in $(L/N^{p+t+1})_L$.*

Proof. $a \in N^p/N^{p+t+1}$ so that $a = [x^p v + N^{p+t+1}]$. $d \in N^p \backslash N^{p+1}$ so that $d = x^p u$ where u is a unit in L. In L, $x^p v u^{-1} = w x^p$ since

$$\begin{aligned} N^p = x^p L = Lx^p. \quad \text{Let } y = [w + N^{p+t+1}].\ yd \\ = [w + N^{p+t+1}]d = [wd + N^{p+t+1}] = [wx^p u + N^{p+t+1}] \\ = [x^p v u^{-1} u + N^{p+t+1}] = [x^p v N^{p+t+1}] = a. \end{aligned}$$

PROPOSITION 2.3. *E_L is isomorphic to the L-injective hull of the simple right L-module $(L/N)_L$.*

Proof. E_L contains a copy of $(L/N)_L$, namely S_1. Thus it is enough to show that E is an essential injective extension of S_1. S_1 is essential in E for if $a \in E$, $a \in S_k$ for some integer k. Let t be the first such integer: then $a \in S_t \backslash S_{t-1}$, a is a generator for S_t, and $aL = S_t$. Thus $aL \cap S_1 = S_1$ and S_1 is essential. Since L is a principal ideal domain, it is a hereditary two-sided order in its quotient division ring. In order to prove E_L is injective it is sufficient by a result of Levy ([5], Theorem 3.4) to show that it is L-divisible. Take $a \in E$ and $0 \neq d \in L$. $a \in S_t$ for some t and $d \in N^p \backslash N^{p+1}$ for some p. $yd = a$ has a solution in S_{p+t+1}, and hence in E, by Lemma 2.2. E is thus an essential injective extension of S_1 and hence is its injective hull.

Let $K = \mathrm{end}_L(E)$ and let K act on E by left multiplication; E then becomes a left K-module. Let $H = \mathrm{end}_K(E)$; in similar manner E then becomes a right H-module. $Ed = E$ (since E is L-divisible) for all nonzero d in L; thus E is a faithful right L-module. Hence L may be considered as a unital subring of H.

LEMMA 2.4. *The S_k's are the only proper L-submodules of E_L.*

Proof. Suppose M_L is a submodule of E with generating set $\{m_i\}$. Since $E = \cup S_k$, each m_i is in some S_k. Let k_i be the first k for which $m_i \in S_k$. Then $m_i \in S_{k_i} \backslash S_{k_i - 1}$ and $m_i L = S_{k_i}$. $M = \Sigma m_i L = \Sigma S_{k_i}$ so that if $\{k_i\}$ is bounded, $M = S_{k_t}$ where $k_t = \max\{k_i\}$, and if $\{k_i\}$ is not bounded, then $M = E_L$.

LEMMA 2.5. *If $a \in S_n$ and if $b \in S_{n-1}$, then there is a $q \in K$ such that $q(b) = a$.*

Proof. Assume that t is the first integer for which $b \in S_{n+t}$. Then $\mathrm{ann}_L(b) = N^{n+t}$ which is contained in N^n which in turn is contained in $\mathrm{ann}_L(a)$. Thus the map $\bar{q}\colon bL \to aL$ defined by $\bar{q}(bd) = ad$ is well defined. E_L is L-injective so that $\bar{q}$ can be extended to an endomorphism q of E. $q \in K$.

PROPOSITION 2.6. *Each S_n is a cyclic left K-submodule of ${}_K E$, the composition length of ${}_K(S_n)$ is n, and the S_n's are the only proper K-submodules of E.*

Proof. If $q \in K, q(S_n)$ is an L-submodule of E of composition length less than or equal to n and hence must be contained in S_n by Lemma 2.4; hence each S_n is a left K-submodule. Each ${}_K(S_n)$ is cyclic via Lemma 2.5; in fact, any L generator of S_n will be a K generator of S_n. This implies that ${}_K(S_1)$ is simple and inductively that the composition length of ${}_K(S_n)$ is n. The proof of Lemma 2.4 shows that these are the only K-submodules of E.

LEMMA 2.7. *Let H_i be the annihilator of S_i in H. Then H_i is a two-sided ideal of H, H_{i+1} is properly contained in H_i, and $\cap H_i = 0$.*

Proof. H_i is clearly a right ideal of H. If $h \in H$, then $(S_i)h$ is a K-submodule of E of composition length less than or equal to i. By Proposition 2.6 it must be that $(S_i)h \subset S_i$ so that each S_i is H-invariant. As a result H_i is a left ideal and hence an ideal. The inclusions are proper, for $H_i \cap L = N^i$ and $N_i \neq N^{i+1}$. Since $E = \cup S_i$, anything in $\cap H_i$ would annihilate all of E and hence be zero.

PROPOSITION 2.8. $H = L$. *That is, L is the second endomorphism ring of E_L.*

Proof. Take $f \in H$. By Proposition 2.6 there is a nonzero $y \in S_1$ such that $S_1 = Ky = yL$. Hence there is *a* $p_1 \in L$ such that $yf = yp_1$. Also, if $z \in S_1$, $z = ky$ for some $k \in K$ and

$$z(f - p_1) = (ky)(f - p_1) = k0 = 0 \text{ . Hence } f - p_1 \in \operatorname{ann}_H(S_1) = H_1 \text{ .}$$

Inductively suppose that there is a $p_i \in L$ such that $f - p_i \in H_i$. Now take $y \in S_{i+1} \backslash S_i$. $y(f - p_i) \in S_{i+1}$ so that there is a $d \in L$ such that $y(f - p_i) = yd$. If $z \in S_{i+1}$, $z = ky$ for some $k \in K$. Then $z(f - p_i) = (ky)(f - p_i) = k(y(f - p_i) = k(yd) = (ky)d = zd$ and hence $f - p_i - d$ is in H_{i+1}. Let $p_{i+1} = p_i + d$; then $f - p_{i+1} \in H_{i+1}$.

The sequence $\{p_i\}$ is Cauchy in L, for $p_n - p_m = (p_n - f) + (f - p_m)$ an element of $H_n + H_m$; but $H_n + H_m = H_n$ if $n \leqq$ m. Thus $p_n = p_m$ is in $H_n \cap L = N^n$. L is complete; therefore $\{p_i\}$ converges to some element p of L. It only remains to be shown that $p = f$. Take $z \in E$; $z \in S_n$ for some n. $\{p_i\}$ converges to p so that there is a positive integer M such that $p_m - p \in N^n$ for all m greater than M. Take m greater than $M + n$. $zf = zp_m = zp$. z was arbitrary; therefore $f = p$.

3. The Ring K.

In this section it will be shown that K is a complete, local, principal ideal domain.

LEMMA 3.1. *Let L, E, and K be as in* §2. *Let J denote the Jacobson radical of K and let $A_n = \operatorname{ann}_K(S_n)$. Then*

(i) *K is a local domain.*

(ii) *$J = A_1$, $J^n \subset A_n \cap A_n = 0$, and $\cap J^n = 0$.*

(iii) *K is complete in the topology induced by the A_n's.*

Proof. (i) K is local since it is the endomorphism ring of an indecomposable injective module. To prove that K is a domain it is sufficient to show that every nonzero endomorphism of E_L is an epimorphism. Let $0 \neq k \in K$. If $k(E) \neq E$, $k(E) = S_n$ for some n by Lemma 2.4. $\operatorname{Ann}_L(S_n) = N^n$; take $0 \neq b \in N^n$. Since E is L-divisible, $Eb = E$. As a result $S_n = k(E) = k(Eb) = k(E)b = S_n b = 0$ contradicting the fact that $k \neq 0$.

(ii) The radical of K, J, is the set of all endomorphisms of E_L whose kernel is essential (see [2], page 44). Since $(S_1)_L$ is the unique minimal submodule of E, $\ker(k)$ is essential if and only if $k(S_1) = 0$;

therefore $J = A_1$ and $JS_1 = 0$. Inductively suppose that $J^{n-1}S_{n-1} = 0$. $JS_n \subset S_{n-1}$ since it is contained in the radical of $K(S_n)$, S_{n-1}. Hence $J^n s_n = J^{n-1}(Js_n)$ which is contained in $J^{n-1}S_{n-1}$ which is zero, hence $J^n \subset A_n \cap A_n = 0$ since anything in $\cap A_n$ would annihilate all of the S_n's and hence all of E. $\cap J^n = 0$ since $J^n \subset A_n$.

(iii) Let $\{f_i\}$ be a Cauchy sequence in K with respect to the topology induced by the decreasing family $\{A_n\}$. Let $x \in E$. $x \in S_p$ for some p. Since $\{f_i\}$ is Cauchy, there is an integer M such that $f_n - f_m \in A_p$ for n, m greater than M. Define $f(x) = f_{M+1}(x)$. It is clear that $f \in K$ and that $f_i \to f$ by the nature of the construction.

Pick $j \in J\backslash A_2$. There is such a j, for if $y_2 \in S_2\backslash S_1$ and if $0 \neq y_1 \in S_1$, then there is a $j \in K$ such that $j(y_2) = y_1$ by Lemma 2.5. $j \in J\backslash A_2$. In fact if $s \in S_{n+1}\backslash S_n$, then $j^n s$ is a nonzero element of S_1. The proof is by induction. If $s \in S_2\backslash S_1$, then $s = y_2 u$ for u a unit in L. Hence $js = jy_2u = y_1u \neq 0$. Inductively suppose that $j^{n-1}s$ is nonzero for all s in $S_n\backslash S_{n-1}$ and take $s \in S_{n+1}\backslash S_n$. $js \in S_n$ by an argument in the previous proof. The claim is that $js \notin S_{n-1}$. If it were, then $j^{n-1}s = 0$ which contradicts the induction hypothesis since $sd \in S_n\backslash S_{n-1}$ for some d in L. Hence $js \notin S_{n-1}$ so again by the induction hypothesis $j^n s = j^{n-1}(js) \neq 0$.

LEMMA 3.2. *Let K, J, j, E, and L be as above.*
(i) $J = jK$.
(ii) $J = Kj$.
(iii) $J^n = j^nK = Kj^n$.

Proof. (i) Let $x \in$ J. Let $y_2 \in S_2\backslash S_1$. $x(y_2) = y \in S_1$ since $x \in J$. Let $j(y_2) = y_1$; y_1 is a nonzero element of S_1 since $j \in J\backslash A_2$. Then there is an element d in L such that $y = y_1 d = j(y_2)d = j(y_2 d)$. By Lemma 2.5 there exists $k_1 \in K$ such that $k_1(y_2) = y_2 d$. If $s \in S_2$, then $s = y_2 c$ for some c in L. $x(s) = x(y_2c) = X(y_2)c = uc = (jk_1(y_2))c = jk_1(y_2c) = jk_1(s)$. This says that $x - jk_1 \in A_2$.

Inductively suppose that there exist $k_1, \cdots, k_{n-1}$ such that

$$\mathrm{z} = x - (jk_1 + j^2k_2 + \cdots + j^{n-1}k_{n-1}) \in \mathrm{A}_n. \quad \text{If}$$
$$y_{n+1} \in S_{n+1}\backslash S_n, \text{ then } j^n(y_{n+1}) = y_1$$

a nonzero element of S_1 by the above choice of j. Also $z(y_{n+1}) \in S_1$ since $z \in A_n$. Hence by the argument above there is a $k_n \in K$ such that $z - j^nk_n \in A_{n+1}$. The sequence $\{jk_1 + \cdots + j^nk_n\}$ converges to x in the A_n topology by the nature of the construction. Also, since $J^n \subset A_n$ the sequence $\{k_1 + \cdots + j^{n-1}k_n\}$ is Cauchy and hence by the completeness of K converges to some element k of K. Also by the construction $jk = x$. Since x was arbitrary in J, $J = jk$.

(ii) is proven by an argument similar to that of (i).

(iii) $J = jK = Kj$ by (i) and (ii). Inductively suppose that $J^n = j^nK = Kj^n$. Then $J^{n+1} = J^nJ = (j^nK)(jK) = j^n(Kj)K = j^n(jK)K = j^{n+1}K$. Similarly $J^{n+1} = Kj^{n+1}$.

PROPOSITION 3.3. *K as above.*

(i) $J^n = A_n$ for all n.

(ii) J^n are the only one-sided ideals of K.

(iii) K is a complete principal ideal domain.

Proof. (i) $J = A_1$ by Lemma 3.1. Inductively suppose that $A_n = J^n$. $J^{n+1} \subset A_{n+1} \subset A_n = J^n$. $J^n/J^{n+1} = j^nK/j^{n+1}K \simeq K/jK = K/J$ which is simple. Therefore either $A_{n+1} = J^{n+1}$ or $A_{n+1} = J^n$. But by the induction hypothesis $j^n \notin A_{n+1}$ so that $A_{n+1} = J^{n+1}$.

(ii) It is sufficient to show that given $x \in K$, $xK = K$ or that $xK = J^p$ for some p. Take $x \in K$ and suppose that $xK \neq K$, then x is not a unit and hence $x \in J^{p+1}$ for some p. By Lemma 3.1 $x = j^pk$, and k must be a unit; for otherwise $k = jk_1$ for some k_1 in K and $x = j^pjk_1 \in J^{p+1}$. As a result $xK = j^pkK = j^pK = J^p$. Similarly $Kx = J^p$.

(iii) K is a principal ideal domain by Lemma 3.2 and (ii). K is complete by (i) and Lemma 3.1.

4. **The Ring R.** Let R, M, $\bar{R}$, and L be as in Theorem 1.1. Then $\bar{R}$ is the full $k \times k$ matrix ring over L. Let e_{ij}, $i, j = 1, 2, \cdots, n$ be a complete set of matrix units for $\bar{R}$. Let M_L be a right L-module and let $M^* = M_1 \oplus \cdots \oplus M_n$, a direct sum of n copies of M. Let f_1 be the identity map on M_1, and let f_i, $i = 2, \cdots, n$ be an isomorphism from M_1 to M_i. Then M^* can be made into an $\bar{R}$-module by defining $f_i(m)e_{ij} = f_j(m)$ and $f_i(m)e_{kj} = 0$ if $i \neq k$. "$*$" is a category isomorphism from the category of right L-modules to the category of right $\bar{R}$-modules. There is also a category isomorphism e_{11} from the category of right $\bar{R}$-modules to the category of right L-modules defined by $(A_{\bar{R}})e_{11} = Ae_{11}$. M and M^*e_{11} are isomorphic for any right L-module M (see [1], or [5] page 137).

PROPOSITION 4.1. *$\bar{R}$ is the second endomorphism ring of the $\bar{R}$-injective hull of the simple right $\bar{R}$-module.*

Proof. Let E be the L-injective hull of the simple right L-module as in §2. Then E^* is the $\bar{R}$-injective hull of *a* simple right $\bar{R}$-module since $*$ is a category isomorphism. $\bar{R}/\bar{M}$ is simple Artinian and $\bar{M}$ is the Jacobson radical of $\bar{R}$ so there is only one isomorphism class of simple right-$\bar{R}$-modules. Let $K = \mathrm{end}_{\bar{R}}(E^*)$ and take $q \in K$.

$q(E^*e_{ii}) = q(E^*{}_{ii}e_{ii}) = q(E^*e_{ii})e_{ii}$; thus each E^*e_{ii} is K-invariant and ${}_KE^* = {}_KF^*e_{11} \oplus {}_KEe_{22} \oplus \cdots \oplus {}_KE^*e_{kk}$. Each e_{ij} is a K-isomorphism so that E^* is decomposed as a direct sum of k mutually isomorphic K-modules. Thus each K-endomorphism of E^* can be given by multiplication by a matrix of homomorphisms. The remainder of the proof shows that the entries in this matrix are of the desired forms. Each $q \in K$ restricted to $E^*{}_{ii}$ is an L-endomorphism of $E^*{}_{ii}$. Each L-endomorphism of E^*e_{ii} can be extended in one and only one way to an $\bar{R}$-endomorphism of E^*; namely, if $\bar{q}$ is an L-endomorphism of E^*e_{ii}, then its unique extension q is defined by $q(z) = \sum_{j=1}^{k}\bar{q}(ze_{ji})e_{ij}$ for $z \in E^*$. Hence $K \simeq \mathrm{end}_L(E^*e_{ii})$ via the restriction map. By proposition 2.8 each element of $\mathrm{end}_K(E^*e_{ii})$ can be given by right multiplication by an element of $e_{ii}\bar{R}e_{ii}$. If $h\colon E^*e_{ii} \to E^*e_{jj}$ is a K-homomorphism, then $h\bar{e}_{ji}$ is a K-endomorphism of E^*e_{ji} where $\bar{e}_{ji}$ denotes right multiplication by e_{ji}. Hence $h\bar{e}_{ji} = \bar{e}_{ii}r\bar{e}_{ii}$ for some $r \in \bar{R}$. If $z \in E^*{}_{ii}$, then $(z)h = zhe_{jj} = zhe_{ji}e_{ij} = ze_{ii}re_{ii}e_{ij} = ze_{ii}re_{ij}$ so that h is given by right multiplication by an element of $e_{ii}\bar{R}e_{ij}$. As a result every K-endomorphism of E^* is given by right multiplication by an element of $\bar{R}$.

R can be considered as a subring of $\bar{R}$; as a result every $\bar{R}$-module is automatically an R-module. Also, if $\bar{M}$ is the maximal two-sided ideal of $\bar{R}$, then $\bar{M}^p \cap R = M^p$ and every coset of $\bar{R}/\bar{M}^p$ has a representative in R (Theorem 1.1).

LEMMA 4.2. *E^* as in the proof of Proposition 4.1, then $(E^*)_R$ is the ascending union of $\bar{R}$-modules $0 \subset B_1 \subset B_2 \subset \cdots$ where the composition length of B_n is n. These are the only $\bar{R}$-submodules of E^*. Furthermore, the B_i's are the only R-submodules of E^* and every R-endomorphism of E^* is an $\bar{R}$-endomorphism. That is, the structure of E^* as an R-module is identical to its structure as an $\bar{R}$-module.*

Proof. The first part follows since it was true of E and $*$ is a category isomorphism. Let $B_i = S_i^*$. A category isomorphism preserves the submodule lattice. Note that the composition length of $(B_n)_{\bar{R}}$ is n; since $\bar{M}$ is the radical of $\bar{R}$, $B_n\bar{M}^n = 0$. In order to prove that the B_n's are the only R-submodules of E^* it is sufficient to show that $aR = a\bar{R}$ for all $a \in E^*$. Take $a \in E^*$. Clearly $aR \subset a\bar{R}$. Take $\bar{r} \in \bar{R}$. $a \in B_n$ for some n so that $a\bar{M}^n = 0$. By theorem 1.1 there is an m in $\bar{M}^n$ so that $\bar{r} + m = r \in R$, then $a\bar{r} = a\bar{r} + 0 = a\bar{r} + am = a(\bar{r} + m)ar$. Thus $a\bar{R} \subset aR$ and $a\bar{R} = aR$.

Let q be an R-endomorphism of E^* and take $a \in E^*$ and $\bar{r} \in \bar{R}$. It must be shown that $q(a\bar{r}) = q(a)\bar{r}$. Since $a \in E^*$, $a \in B_n$ for some n. The B_n's are the only R-submodules of E^* and the composition length of B_n is n, so that $q(B_n) \subset B_n$ and $q(a) \in B_n$. As above there is an $m \in \bar{M}^n$ such that $\bar{r} + m = r \in R$. $B_n\bar{M}^n = 0$. Then

$$\begin{aligned} q(a\bar{r}) &= q(a\bar{r} + 0) = q(a\bar{r} + am) = q(a(\bar{r} + m)) = q(ar) \\ &= q(a)r = q(a)(\bar{r} + m) = q(a)\bar{r} + q(a)m \\ &= q(a)\bar{r} + 0 = q(a)\bar{r} \;. \end{aligned}$$

Thus q is an $\bar{R}$-endomorphism.

LEMMA 4.3 *E^* is the R-injective hull of $(B_1)_R$.*

Proof. By Lemma 4.2 $(B_1)_R$ is an essential submodule of $E^*{}_R$. E^* is an injective $\bar{R}$-module since * is a category isomorphism; in particular E^* is a divisible $\bar{R}$-module so that E^* is a divisible R-module. R is a hereditary two-sided order so that E^* is an injective R-module by [5], Theorem 3.4.

THEOREM 4.4. *Let R be a Dedekind prime ring with a maximal two-sided ideal M, and let P be a maximal right ideal of R containing M. Then the R-endomorphism ring of the R-injective hull of R/P is a complete principal ideal domain.*

Proof. Let R, $\bar{R}$, L, E_L, and E^* be as above. Then by Lemma 4.3 E^* is the injective hull of a simple right R-module which is annihilated by M. $(B_1)_R \simeq R \backslash P$ since both are simple modules over the simple Artinian ring R/M; thus $E^* \simeq E(R/P)$. By Lemma 4.2 $\mathrm{end}_R(E^*) = \mathrm{end}_{\bar{R}}(E^*)$ which is isomorphic to $\mathrm{end}_L(E)$ since* is a category isomorphism. Hence the result follows by Proposition 3.3.

THEOREM 4.5. (Main Theorem) *Let R be a Dedekind prime ring with a nonzero prime ideal M, and let P be a maximal right ideal containing M with $E(R/P)$ the R-injective hull of R/P. Then $\bar{R}$, the completion of R at M, is isomorphic to the second endomorphism ring of $E(R/P)$.*

Proof. Consider E^*; as above $E^* \simeq E(R/P)$. By Lemma 4.2 the R and $\bar{R}$ structures of E^* are identical. Thus $\bar{R}$ is second endomorphism ring of $E(R/P)$ by Proposition 4.1.

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