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THE p-PARTS OF CHARACTER DEGREES IN p-SOLVABLE GROUPS

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Let G be a finite group and $Irr(G)$ the set of irreducible complex characters of G. Fix a prime integer p and let $e(G)$ be the largest integer such that $p^{e(G)}$ divides $\chi(1)$ for some $\chi \in Irr(G)$. The purpose of this paper is to obtain information about the structure of G, and in particular about a Sylow p-subgroup of G, from a knowledge of $e(G)$. If G is solvable, we obtain the bound $2e(G)+1$ for the derived length of an S_p subgroup of G. We also obtain some information about the normal structure of G in terms of $e(G)$.

When $e(G) = 0$, our result is equivalent to the theorem of N. Ito which asserts that G has a normal abelian Sylow p-subgroup. Actually, Ito's result, [7], holds for p-solvable groups. This may readily be proved by induction on the group order, as follows. The hypothesis $e(G) = 0$ is inherited by factor groups and by normal subgroups and it follows easily that a minimal counterexample, G, has a normal p-complement, H. Now let $\chi \in Irr(G)$. It follows from Clifford's theorem that $t|\chi(1)$, where t is the index in G of the inertia group of an irreducible constituent of the restriction χ_H. Since t is a power of p, we have $t = 1$, and every irreducible constituent of χ_H is invariant in G. It follows by Frobenius reciprocity that every irreducible character of H is invariant in G. Now Lemma 2.1 of [4] applies to yield the result.

Although it might be conjectured that our present bounds hold for all p-solvable groups when $e(G) > 0$, the proofs given here fail even when $e(G) = 1$. However in this case, we do obtain a result which is valid for p-solvable G with $p > 3$, and shows that $\mathcal{O}_p(G)$ is either abelian or else is a Sylow subgroup of G.

1. The following lemma is well known and will be used repeatedly. Since its proof is quite short, we present it here.

LEMMA 1.1. *Let $N \lhd G$ and $\chi \in Irr(G)$. Suppose θ is an irreducible constituent of χ_N. Let $T = \mathscr{I}_G(\theta)$, the inertia group of θ. Then there exists a unique irreducible constituent ψ of χ_T such that θ is a constituent of ψ_N. Furthermore $\chi = \psi^G$ and $[\chi_N, \theta] = [\psi_N, \theta]$.*

Proof. Choose any irreducible constituent ψ *of* χ_T such that θ is a constituent of ψ_N. By Clifford's theorem, $\chi_N = a\sum_{i=1}^{t} \theta_i$ where $\theta_1 = \theta$

and $\chi(1) = at\theta(1)$. We have $t = |G: T|$ and $\psi_N = a_0\theta,\ a_0 \leqq a$. Now χ is a constituent of ψ^G and so

$$\chi(1) \leqq \psi^G(1) = t\psi(1) = ta_0\theta(1) \leqq ta\theta(1) = \chi(1) .$$

We have equality throughout, so that $\chi(1) = \psi^G(1)$ and $a = a_0$. Thus $\chi = \psi^G$ and $[\chi_N, \theta] = a = a_0 = [\psi_N, \theta]$. The uniqueness of ψ also follows from $a = a_0$.

If $e(G) = e$ and $N \lhd G$, let $\theta \in Irr(N)$ and $T = \mathscr{I}_G(\theta)$. Suppose that $|G: T|_p = p^r$, where n_p denotes the p-part of the integer n. Let ψ be any irreducible constituent of θ^T, and let χ be an irreducible constituent of ψ^G. Then by Frobenius reciprocity and Lemma 1.1, it follows that $\chi = \psi^G$ and hence $\psi(1)_p \leqq p^{e-r}$. It does not follow, however, that $e(T) \leqq e - r$. We wish to prove our results by induction in a manner similar to this and hence we define a quantity which "inducts" properly.

DEFINITION 1.2. Let $N \lhd G$ and $\theta \in Irr(N)$. Suppose θ is invariant in G. Then $e(G, N, \theta) = e$ is the largest integer such that $p^e | (\chi(1)/\theta(1))$ for some irreducible constituent χ of θ^G.

Note that $e(G, 1, 1) = e(G)$ and that if $N \subseteqq H \lhd G$, then

$$e(H, N, \theta) \leqq e(G, N, \theta) .$$

The following is immediate.

COROLLARY 1.3. *Suppose $e(G, N, \theta) = e$ and $N \subseteqq M \lhd G$. Let ψ be an irreducible constituent of θ^M and let $p^f = (\psi(1)/\theta(1))_p$. Set $T = \mathscr{I}_G(\psi)$ and $p^r = |G: T|_p$. Then $e(T, M, \psi) \leqq e - f - r$.*

It would suffice for our purposes to show that if $N \lhd G$, G/N is solvable and $e(G, N, \theta) = e$ for some $\theta \in Irr(N)$, then the derived length of an S_p subgroup of G/N is bounded by a function of e. We in fact will prove this for certain special characters θ and also for certain groups G/N. In order to prove results like these, it is necessary to be able to produce irreducible characters of degrees divisible by "large" powers of p. This is done using the following result of Gallagher ([1], Theorem 2).

PROPOSITION 1.4. *Let $N \lhd G$ and suppose $\chi \in Irr(G)$ and*

$$\chi_N = \theta \in Irr(N) .$$

Then the irreducible constituents of θ^G are uniquely of the form $\beta\chi$ where $\beta \in Irr(G/N)$ is viewed as a character of G. For all such β,

$\beta\chi$ is irreducible.

LEMMA 1.5. *Let $N \lhd G$, $N \subseteqq H \lhd G$ with G/H a p-group. Let $\theta \in Irr(N)$ be invariant in G. If $e(G, N, \theta) = e(H, N, \theta)$, then G/H is abelian. If $e(G, N, \theta) > e(H, N, \theta)$, then there exists $L \lhd G$ with $H \subseteqq L$, G/L abelian and $e(L, N, \theta) < e(G, N, \theta)$.*

Proof. Let $K \lhd G$, $K \supseteqq H$ be minimal such that

$$e(K, N, \theta) = e(G, N, \theta) = e\ .$$

Let ψ be an irreducible constituent of θ^K with $p^e|(\psi(1)/\theta(1))$. Let χ be any irreducible constituent of ψ^G. Then $p^{e+1} \nmid (\chi(1)/\theta(1))$ and therefore $p \nmid (\chi(1)/\psi(1))$. Since G/K is a p-group, $\chi(1)/\psi(1)$ is a power of p and thus $\chi(1) = \psi(1)$ and $\chi_K = \psi \in Irr(K)$. Let β be an arbitrary irreducible character of G/K. By Proposition 1.4, $\beta\chi$ is an irreducible constituent of ψ^G and we may apply the above reasoning to $\beta\chi$ in place of χ. Hence $(\beta\chi)(1) = \psi(1) = \chi(1)$ and $\beta(1) = 1$. Thus G/K is abelian. If $e(G, N, \theta) = e(H, N, \theta)$ then $H = K$ and the first statement is proved.

Otherwise $K > H$ and we may choose $L \lhd G$ with $H \subseteqq L < K$ and $|K: L| = p$. By the choice of K, $e(L, N, \theta) < e$ and hence ψ_L is reducible. Therefore $\chi_L = \psi_L$ is a sum of p distinct irreducible characters, conjugate in K. Let φ be one of these characters and put $T = \mathscr{I}_G(\varphi)$ so $|G: T| = p$. Thus $T \lhd G$ and $G' \subseteqq T$. We also have $G' \subseteqq K$ and $K \cap T = L$ so that G/L is abelian and the result follows.

LEMMA 1.6. *Let $N \lhd G$ and suppose that G/N is p-solvable with p'-length $\leqq 1$. Suppose $\theta \in Irr(N)$ and is invariant in G with*

$$e(G, N, \theta) = e\ .$$

Then the derived length of an S_p subgroup of G/N is $\leqq e + 2$. If G/N is a p-group, d.l.$(G/N) \leqq e + 1$.

Proof. Let $K/N = \mathscr{O}^p(G/N)$, the minimum normal subgroup with factor group a p-group. By hypothesis, K/N has the normal S_p subgroup P/N. Suppose $e(K, N, \theta) < e$. Then by Lemma 1.5, there exists $L \lhd G$, $K \subseteqq L$ with G/L abelian and $e(L, N, \theta) < e$. Both statements now follow by induction on $|G: N|$. Suppose then $e(K, N, \theta) = e$. Then G/K is abelian by Lemma 1.5. If $K = N$, then d.l.$(G/N) \leqq e + 1$ is trivial. Suppose, then, $K > N$. Then $P < K$ and $e(P, N, \theta) \leqq e$ so by induction, d.l.$(P/N) \leqq e + 1$. Since G/K is abelian, the derived length of an S_p subgroup of G/N is $\leqq e+2$. However, since $K>N$, G/N is not a p-group and the proof is complete.

2. Suppose $N \lhd G$ and $\theta \in Irr(N)$ and is invariant in G. It will occasionally be necessary in what follows to be able to extend θ to an irreducible character of G. This is, of course, not always possible. We discuss some sufficient conditions below.

Given any character χ of a finite group G, we define the determinant $\det \chi = \lambda$ to be the linear character of G given by

$$\lambda(g) = \det \mathfrak{X}(g) ,$$

where $\mathfrak{X}$ is any representation affording χ. Let $o(\chi)$ denote the order of λ as an element of the group of linear characters of G. Clearly $o(\chi) = o(\lambda) = |G\colon \ker \lambda|$. Gallagher [1] has shown that if $\theta \in Irr(N)$, $N \lhd G$, θ is invariant G and $(\theta(1), |G\colon N|) = 1$, then θ is extendible to G if and only if $\det \theta$ is extendible to G. Furthermore, Gallagher proved that if $\lambda = \det \theta$ and μ is an extension of λ, then there is a unique extension χ of θ with $\det \chi = \mu$. Since θ is invariant in G, so is λ and it follows that $\ker \lambda \lhd G$ and $N/\ker \lambda \subseteqq \mathbf{Z}(G/\ker \lambda)$. If $(o(\theta), |G\colon N|) = 1$, then $N/\ker \lambda$ is a direct factor of $G/\ker \lambda$ and hence there is a unique extension μ of λ to G with $o(\mu) = o(\lambda)$. Summarizing these results, we obtain the following.

PROPOSITION 2.1. *Let $N \lhd G$ and let $\theta \in Irr(N)$ with θ invariant in G. Suppose $o(\theta)$ and $\theta(1)$ are both relatively prime to $|G\colon N|$. Then there exists a unique extension, $\hat{\theta}$, of θ to G with $o(\hat{\theta}) = o(\theta)$.*

DEFINITION 2.2. Let $\chi \in Irr(G)$. Then χ is a *p-character* of G if $\chi(1)$ and $o(\chi)$ are powers of p.

LEMMA 2.3. *Let $N \lhd G$ and suppose $\theta \in Irr(N)$ is a p-character which is invariant in G. Suppose G/N has a normal p-complement K/N and that $\mathcal{O}_p(G/N) = 1$. Then* $\mathrm{d.l.}(G/K) \leqq e(G, N, \theta) = e$.

Proof. Use induction on $|G\colon N|$. Suppose $e>0$. If $e(K, N, \theta) = e$, then by Lemma 1.5, G/K is abelian and we are done. Otherwise, $e(L, N, \theta) < e$ for some $L \lhd G$ with $K \subseteqq L$ and G/L abelian. By induction, $\mathrm{d.l.}(L/K) \leqq e-1$ and the result follows. The only remaining case is where $e = 0$. Here we must show that $K = G$.

Since θ is a p-character of N, there is an extension $\hat{\theta}$ of θ to K. Let χ be any irreducible constituent of $\hat{\theta}^G$. Then $\chi(1)/\theta(1)$ is a power of p and θ is a constituent of χ_N so $\chi(1) = \theta(1)$ since $e(G, N, \theta) = 0$. Thus if β is any irreducible character of G/N, $\beta\chi \in Irr(G)$ and since θ is a constituent of $(\beta\chi)_N$, it follows that $p \nmid \beta(1)$. Hence $e(G/N) = 0$

and therefore G/N has a normal S_p subgroup. Since $\mathcal{O}_p(G/N) = 1$, $p \nmid |G: N|$ and thus $K = G$ and the proof is complete.

The following lemma will be used to prove that a given character is a p-character.

LEMMA 2.4. *Let $N \lhd G$ and suppose that G/N has no proper normal subgroup of p'-index. Let $\chi \in Irr(G)$ and suppose θ is an irreducible constituent of χ_N and $o(\theta)$ is a power of p. Then* $o(\chi)$ is *a power of p.*

Proof. Let $\lambda = \det \chi$, and let $K = \{g \in G \mid \lambda(g)^{p^e} = 1$ for some $e \geqq 0\}$. It suffices to show that $K = G$. Clearly, $K \lhd G$ is a subgroup, and $p \nmid |G: K|$. The result will follow if we show $N \subseteqq K$. Now $\chi_N = a\Sigma\theta_i$ by Clifford's theorem, where the θ_i are all conjugate to θ. Let $\mu_i = \det \theta_i$ so that $\lambda_N = (\Pi\mu_i)^a$. Each μ_i has order equal to $o(\theta)$ which is a power of p. Therefore, for suitable e, and for $x \in N$, we have $\mu_i(x)$ is a p^e-th root of 1. It follows that $N \subseteqq K$ and the proof is complete.

3. We define functions u, v as follows.

DEFINITION 3.1. Let u, v be functions from the set of nonnegative integers into the same set with ∞ adjoined, where $u(e)$ = maximum derived length of an S_p subgroup of G/N where G is a finite group, $N \lhd G$, G/N is solvable and there exists a p-character, θ, of N, invariant in G and such that $e(G, N, \theta) \leqq e$. Set $u(e) = \infty$ if there is no maximum. Define $v(e)$ similarly, except that only those situations are considered where $\mathcal{O}_p(G/N) = 1$.

LEMMA 3.2. *Let P be a p-group and suppose that $P_0 \subseteqq P$ with $|P: P_0| = p^r$. Then* d.l.$(P) \leqq r +$ d.l.(P_0).

Proof. Use induction on r. The result is trivial if $r = 0$. Otherwise $P_0 < P$ and hence $P_0P' < P$ since $P' \subseteqq \Phi(P)$, the Frattini subgroup of P. By induction, d.l.$(P_0P') \leqq (r - 1) +$ d.l.(P_0). However, $P_0P' \lhd P$ and P/P_0P' is abelian. The result follows.

LEMMA 3.3. *Let $N \subseteqq H$ be normal subgroups of L. Assume $(|H: N|, |L: H|) = 1$. Let $\theta \in Irr(N)$ and suppose θ is extendible to H. If θ is invariant in L, then some extension of θ to H is also invariant in L.*

Proof. Let $\mathcal{S}$ be the set of extensions of θ to H, and let U be the group of linear characters of H/N. Then U acts on the set $\mathcal{S}$

by multiplication and by Proposition 1.4, this action is transitive. Set $A = L/H$. We have $U \cong H/H'N$ and thus $(|A|, |U|) = 1$. Clearly, A acts on $\mathscr{S}$ and on the group U and if $\chi \in \mathscr{S}$, $\lambda \in U$, then $(\chi\lambda)^a = \chi^a\lambda^a$ for all $a \in A$. Therefore Glauberman's Lemma (Theorem 4 of [2]) applies and hence A fixes some $\chi \in \mathscr{S}$. Thus χ is invariant in L.

Before going on to our main result, we digress briefly to give an application of some of the lemmas we have already accumulated.

COROLLARY 3.4. *Let $N \lhd G$ with G/N p-solvable. Suppose θ is a p-character of N which is invariant in G and that $e(G, N, \theta) = 0$. Then θ is extendible to G and G/N has a normal abelian S_p subgroup.*

Proof. If θ is extendible to G, then it follows from Proposition 1.4 that $e(G/N) = 0$ and hence G/N has a normal abelian S_p subgroup. We prove extendibility by induction on $|G: N|$. Let $M/N = \mathscr{O}^p(G/N)$. If $M < G$, then θ is extendible to $\psi \in Irr(M)$. Let χ be any irreducible constituent of ψ^G. Since G/M is a p-group, it follows that $\chi(1)/\psi(1)$ is a power of p. Since $e(G, N, \theta) = 0$, $\chi(1) = \psi(1)$ and the result follows.

Suppose then $M = G$ and let $V/N = \mathscr{O}^{p'}(G/N)$. Then $V < G$ and θ is extendible to V. Let $W/N = (V/N)'$. Then V/W is a p-group. Now if $x \in G$, then ψ^x is an extension of θ so $\psi^x = \lambda\psi$ for some linear character λ of G/N (Proposition 1.4). Then $\lambda_W = 1$ and $\psi^x_W = \psi_W$. Hence ψ_W is invariant in G and by Lemma 3.3 we may assume that ψ is invariant in G. By Lemma 2.4, ψ is a p-character of V and thus is extendible to G. The proof is complete.

THEOREM 3.5. *The functions u and v are finite valued, $v(0) = 0$, $u(0) = 1$ and*

$$v(e) \leqq \max_{0<f\leqq e} (f + u(e-f)) \text{ for } e > 0 \text{ and}$$

$$u(e) \leqq 1 + \max_{0<f\leqq e} (f + u(e-f)) \text{ for } e > 0 \,.$$

Proof. If u ever takes on the value ∞, choose $e \geqq 0$ minimal with $u(e) = \infty$. Otherwise pick e arbitrarily. Choose a group G, $N \lhd G$, θ a p-character of N, invariant in G with $e(G, N, \theta) \leqq e$. Let P/N be an S_p subgroup of G/N. If $e > 0$, write $b = \max\{f + u(e-f) \mid 0 < f \leqq e\}$. If $e = 0$, set $b = 0$. We claim that (a) if $\mathscr{O}_p(G/N) = 1$, then d.l.$(P/N) \leqq b$ and in any case (b) d.l.$(P/N) \leqq b + 1$. The proof will be complete when these claims are established. In particular, the inequality involving $v(e)$ will follow when (a) is proved. Note that when $e = 0$, the result follows from Corollary 3.4, however this case also follows from the general argument and we do not appeal to the previous result. We shall prove (a) and (b) by induction on $|G: N|$, for the fixed value of e chosen above.

Case 1. $\mathcal{O}_{p'}(G/N) > 1$. Let K/N be a minimal normal p'-subgroup of G/N so that K/N is an elementary abelian q-group for some $q \neq p$. Let $\hat{\theta}$ be the unique extension of θ to K with $o(\hat{\theta}) = o(\theta)$. Because of the uniqueness, $\hat{\theta}$ is invariant in G and by definition, $\hat{\theta}$ is a p-character. Clearly $e(G, K, \hat{\theta}) \leqq e$ and thus d.l.$(PK/K) \leqq b + 1$ by induction. Since $PK/K \cong P/N$, (b) follows in this case. If $\mathcal{O}_p(G/K) = 1$, then d.l.$(PK/K) \leqq b$ and (a) follows.

Assume that $\mathcal{O}_p(G/N) = 1$ but that $\mathcal{O}_p(G/K) = H/K > 1$. Let ψ be an irreducible constituent of θ^H with $(\psi(1)/\theta(1))_p = p^f$ as large as possible. Let φ be an irreducible constituent of ψ_K which is a constituent of θ^K. Since K/N is abelian, it follows from Proposition 1.4 that $\varphi = \hat{\theta}\lambda$ for a linear character λ of K/N. Thus $\varphi(1) = \theta(1)$ is a power of p. Since H/K is a p-group, $\psi(1)/\varphi(1)$ is a power of p and hence $\psi(1)$ is a power of p. We claim that ψ is a p-character of H. This will follow from Lemma 2.4 when we establish that H/N has no nontrivial p'-factor group.

Now $H'N \cap K \lhd G$ and by the minimality of K, we have either $H'N \cap K = N$ or $H'N \cap K = K$. In the first situation, $K/N \subseteqq \mathbf{Z}(H/N)$ and it follows that $\mathcal{O}_p(H/N) \cong H/K > 1$, a contradiction. Thus $H'N \cap K = K$. Since any p'-factor group of H/N is abelian, this shows that only the trivial one exists.

Let $T = \mathscr{I}_G(\psi)$ and set $p^r = |G: T|_p$. By Corollary 1.3,

$$e(T, H, \psi) \leqq e - f - r \,.$$

Let P_0/H be an S_p subgroup of T/H and assume that $P_0 \subseteqq PK$ since PK/H is an S_p subgroup of G/H. Now d.l.$(P_0/H) \leqq u(e - f - r)$ and $|PK: P_0| = p^r$ so that d.l.$(PK/H) \leqq r + u(e - f - r)$ by Lemma 3.2. We have $e(H, N, \theta) = f$ and $\mathcal{O}_p(H/N) = 1$ and hence $0 <$ d.l.$(H/K) \leqq f \leqq e$ by Lemma 2.3. It follows that d.l.$(PK/K) \leqq r + f + u(e - f - r) \leqq b$ and the proof of Case 1 is complete. In particular, since only Case 1 can occur when $\mathcal{O}_p(G/N) = 1$, we have shown that $v(e) \leqq b$.

Case 2. $\mathcal{O}_{p'}(G/N) = 1$. Let $H/N = \mathcal{O}_p(G/N) > 1$ and let

$$f = e(H, N, \theta) \,.$$

Since H/N is a p-group, we can pick $\psi \in Irr(H)$ with $\psi_N = p^f\theta$. Also ψ is a p-character by Lemma 2.4. Let $T = \mathscr{I}_G(\psi)$ and $p^r = |G: T|_p$. Reasoning exactly as before, we get

$$\text{d.l.}(P/N) \leqq r + u(e - f - r) + \text{d.l.}(H/N) \,.$$

By Lemma 1.6, d.l.$(H/N) \leqq f + 1$, and thus

$$\text{d.l.}(P/N) \leqq 1 + f + r + u(e - f - r) \,.$$

If $f + r > 0$, then $e > 0$ and we obtain d.l.$(P/N) \leqq b + 1$ and we are done in this case.

Assume then that $f = 0 = r$ for all irreducible constituents ψ of θ^H. From $f = 0$, it follows that θ is extendible to H and by Lemma 3.3, we may choose an extension ψ which is invariant in L where $L/H = \mathcal{O}_{p'}(G/H)$. Now let $T = \mathcal{I}_G(\psi)$. Since $r = 0$, we may assume $P \subseteq T$. Also $L \subseteq T$. We claim that $U/H = \mathcal{O}_p(T/H) = 1$. We have $[L, U] \subseteq H$ and hence by Lemma 1.2.3 of [3], it follows that $U \subseteq H$. Therefore, d.l.$(P/H) \leqq v(e)$ since $e(T, H, \psi) \leqq e$. By Lemma 1.6, d.l.$(H/N) \leqq 1$ and thus d.l.$(P/N) \leqq 1 + v(e)$. Since we have already shown that $v(e) \leqq b$, the result follows.

COROLLARY 3.6. *$v(e) \leqq 2e$, $u(e) \leqq 2e + 1$ for all $e \geqq 0$.*

Proof. Use induction on e. The Corollary is immediate if $e = 0$. For $e > 0$ we have $v(e) \leqq \max\{f + u(e - f) \mid 0 < f \leqq e\} \leqq \max\{f + 2(e - f) + 1\}$. This maximum occurs when $f = 1$ and yields $v(e) \leqq 2e$. Similarly $u(e) \leqq 2e + 1$.

4. Some improvement on the bounds of Theorem 3.5 can be obtained, especially for $e < p - 1$. We shall use Theorem B of Hall and Higman [3] and also the following result of Passman (Corollary 2.4 (i) of [8]).

PROPOSITION 4.1. *Let P be a p-group which acts faithfully on a solvable p'-group A. Suppose that every element of A lies in an orbit of size $\leqq p^e < p^p$ under the action of P. Then some element of A lies in a regular orbit and hence $|P| \leqq p^e$.*

LEMMA 4.2. *Let $N \subseteq H$ be normal subgroups of L. Suppose H/N is solvable and that $(|L : H|, |H : N|) = 1$. Let $\theta \in Irr(N)$ and suppose $\mathcal{I}_L(\theta)$ covers L over H. Then some irreducible constituent ψ of θ^H is invariant in L.*

Proof. We use induction on $|H : N|$. The result is trivial if $N = H$. Let $M \lhd L$, $M < H$ be maximal such. By the Schur-Zassenhaus Theorem, applied to the group $\mathcal{I}_L(\theta)/N$ which has the normal Hall subgroup, $\mathcal{I}_H(\theta)/N$, we can find a subgroup $S \subseteq L$ with $S \cap H = N$, $SH = L$ and $S \subseteq \mathcal{I}_L(\theta)$. By induction applied to the situation $N \lhd M \lhd SM$, there exists an irreducible constituent φ of θ^M which is invariant under S. Since H/M is an elementary abelian chief factor of L, Proposition 3, Part 2 of [5] applies and we conclude that there are only three cases to consider. They are (a) $\varphi^H = \psi$ is irreducible, (b) $\varphi^H = a\psi$ where ψ is irreducible or (c) φ is extendible to H. In either of cases (a) or (b), ψ is invariant under S and since $L = HS$, we are done. In the remaining case, φ is invariant in L and the result follows from Lemma 3.3.

We state below the special case of Theorem B of Hall and Higman which will be needed in what follows.

PROPOSITION 4.3. *Let G be a p-solvable group which acts faithfully and irreducibly on an elementary abelian p-group U. Suppose $|U| < p^{p-1}$. Then $p \nmid |G|$.*

THEOREM 4.4. *Let $e < p - 1$. Then $u(e) \leqq e + 2$ and $v(e) \leqq e$. If $e(G, N, \theta) < p - 1$, where θ is p-character and G/N is solvable, then $G/N = \mathcal{O}_{pp'pp'}(G/N)$.*

Proof. The first statement follows from the second by Lemmas 1.6 and 2.3 since in calculating $u(e)$ and $v(e)$, it is sufficient to consider only cases where $G/N = \mathcal{O}^{p'}(G/N)$. We proceed to prove the second statement.

Let $N \lhd G$, θ an invariant p-character of N and

$$e(G, N, \theta) = e < p - 1 .$$

It suffices to show that $\mathcal{O}^{p'pp'p}(G/N) = 1$ and this is done by induction on $|G: N|$. If $\mathcal{O}^{p'}(G/N) = L/N$ and $L < G$, then since

$$e(L, N, \theta) \leqq e(G, N, \theta) ,$$

the result follows by induction. Thus we may assume that

$$\mathcal{O}^{p'}(G/N) = G/N$$

and similarly, $\mathcal{O}^{pp'pp'p}(G/N) = 1$. Let $H/N = \mathcal{O}^{pp'pp'}(G/N)$ and $U/N = \mathcal{O}^{pp'p}(G/N)$ so that U/N has the normal S_p subgroup H/N. We may assume $U > N$. Let $V/N = \mathcal{O}^{pp'}(G/N)$ so that V/U is a p-group. Suppose $U \subseteqq Y < V$ with $Y \lhd G$ and $|V: Y| < p^{p-1}$. Let Y be a maximal such subgroup. Then V/Y is an elementary abelian p-group which is an irreducible G/V module. Let $C/V = \mathbf{C}_{G/V}(V/Y)$ so V/Y is a faithful irreducible G/C module. By Proposition 4.3, G/C is a p'-group and since $G/N = \mathcal{O}^{p'}(G/N)$, we have $C = G$. It follows that V/Y is a direct factor of M/Y where $M/N = \mathcal{O}^{p}(G/N)$. Since $V/Y > 1$, this contradicts $\mathcal{O}^{p}(M/N) = M/N$ and therefore no such Y exists.

Now let $U_0/H = (U/H)'$ and let $Y/U = \mathbf{C}_{V/U}(U/U_0)$. Then $Y \lhd G$. Now $U/U_0 \subseteqq \mathbf{Z}(Y/U_0)$ and U/U_0 is a nontrivial $S_{p'}$ subgroup of Y/U_0 since $U > H$ and U/H is a solvable p'-group. It follows that $\mathcal{O}^{p'}(Y/U_0) < Y/U_0$. Since $\mathcal{O}^{p'}(V/N) = V/N$, it must be that $Y < V$.

We will have the desired contradiction when we show $|V: Y| \leqq p^e < p^{p-1}$.

By Lemma 4.2, there exists an irreducible constituent ψ of θ^H such that ψ is invariant in U. Since H/N is a p-group, it follows from Lemma 2.4 that ψ is a p-character of H and hence there exists a unique extension $\hat{\psi}$ of ψ to U with $o(\psi) = o(\hat{\psi})$. It follows from the uniqueness that $\mathscr{I}_G(\psi) = \mathscr{I}_G(\hat{\psi})$. Now let λ be any linear character of U/H. Then $\lambda\hat{\psi} \in Irr(U)$. Let $T = \mathscr{I}_G(\lambda\hat{\psi})$ and put $|G: T| = p^r$. By Corollary 1.3, $e(T, U, \lambda\hat{\psi}) \leqq e - r$ and thus $r \leqq e$. Let $x \in T$. We have

$$\lambda\hat{\psi} = (\lambda\hat{\psi})^x = \lambda^x\hat{\psi}^x .$$

Restricting this to H, we obtain $\psi = \psi^x$ since $\lambda_H = 1$ and $\hat{\psi}_H = \psi$. Thus $x \in \mathscr{I}(\psi) = \mathscr{I}(\hat{\psi})$. Therefore $\lambda\hat{\psi} = \lambda^x\hat{\psi}$ and it follows from Proposition 1.4 that $\lambda = \lambda^x$. Thus $T \subseteqq \mathscr{I}_G(\lambda)$. Since $|G: T|_p = p^r$ and V/U is a normal p-subgroup of G/U, it follows that $|V: T \cap V| \leqq p^r$. Thus $|V: \mathscr{I}_V(\lambda)| \leqq p^r \leqq p^e < p^p$. Therefore, in the action of the p-group V/U on the group of linear characters of U/H, all orbits have size $\leqq p^e$. The kernel of this action is Y/U and thus by proposition 4.1, $|V/Y| \leqq p^e$ which yields the desired contradiction and the proof is complete.

COROLLARY 4.5. *If* $e \geqq p - 1$, $u(e) \leqq 2e - p + 4$ *and*

$$v(e) \leqq 2e - p + 3 .$$

Proof.

$$\begin{aligned} u(p-1) &\leqq \max_{0<f\leqq p-1} \{u(p-1-f) + f\} + 1 \\ &\leqq \max_{0<f\leqq p-1} \{p - 1 - f + 2 + f\} + 1 = p + 2 \end{aligned}$$

and similarly $v(p-1) \leqq p + 1$. Thus the desired inequalities hold when $e = p - 1$. For $e > p - 1$, apply induction.

5. In this section we consider the case $e = 1$ in more detail. From Theorem 4.4 we have $u(1) \leqq 3$ and $v(1) \leqq 1$ when $p \geqq 3$. For $p = 2$, Corollary 3.6 yields $u(1) \leqq 3$ and $v(1) \leqq 2$. An example (see 6.1) shows that $u(1) = 3$ for $p = 3$.

THEOREM 5.1. *For all primes,* $v(1) = 1$.

Proof. That $v(1) \geqq 1$ is clear. Let $e(G, N, \theta) = 1$ with G/N solvable and θ an invariant p-character. Suppose $\mathscr{O}_p(G/N) = 1$. We must show that an S_p subgroup, P/N, of G/N is abelian. Let K/N be a minimal normal subgroup of G/N so that K/N is an elementary abelian q-group for some prime $q \neq p$. Let $\hat{\theta}$ be the unique extension of θ to K with $o(\hat{\theta}) = o(\theta)$. Then $\hat{\theta}$ is an invariant p-character of K. If $\mathcal{O}_p(G/K) = 1$, then the result follows by induction on $|G: N|$. Assume then that $H/K = \mathcal{O}_p(G/K) > 1$. Let λ be any linear character of H/K. Then $\mathcal{I}_G(\lambda) = \mathcal{I}_G(\hat{\theta}\lambda)$ and thus $p^2 \nmid |G: \mathcal{I}_G(\lambda)|$. It follows that λ lies in an orbit of size 1 or p under the action of H/K on the group of linear characters of K/N. Since $\mathcal{O}_p(G/N) = 1$, $\mathbf{C}_{H/K}(K/N) = 1$ and thus H/K acts faithfully on the linear characters of K/N. By Proposition 4.1, $|H/K| = p$.

Now choose λ as above in an orbit of size p. Then

$$(\lambda\hat{\theta})^H = \psi \in Irr(H)$$

and ψ is a p-character of H by Lemma 2.4 (using the minimality of K). Let $T = \mathcal{I}_G(\psi)$ and $T_0 = \mathcal{I}_G(\lambda\hat{\theta})$ so that $HT_0 \subseteq T$ and $T_0 \cap H = K$. By the usual argument, $p^2 \nmid |G: T_0|$ and hence $p \nmid |G: HT_0|$ and we may assume that $P \subseteq HT_0$. Then $PK/K = (H/K)(P_0/K)$ where $P_0 = PK \cap T_0$. Now $e(T, H, \psi) = 0$ by Corollary 1.3 and since $u(0) = 1$, we have PK/H is abelian. But $PK/H \cong P_0/K$ and $H/K \subseteq \mathbf{Z}(PK/K)$ and thus $PK/K \cong P/N$ is abelian. The proof is complete.

We now prove a result which is valid for p-solvable groups with $p > 3$. It will enable us to conclude for solvable groups that $u(1) \leqq 2$ with respect to these primes.

THEOREM 5.2. *Let $N \lhd G$ with G/N p-solvable and $p > 3$. Suppose θ is a p-character of N which is invariant in G and that $e(G, N, \theta) = 1$. Let $P/N = \mathcal{O}_p(G/N)$ and suppose that P/N is not abelian. Then P/N is an S_p subgroup of G/N.*

Proof. Use induction on $|G: N|$ and assume that $P/N \notin Syl_p(G/N)$. Then P/N is a Sylow subgroup of every proper normal subgroup of G/N which contains it. It follows that $\mathcal{O}^{p'}(G/N) = G/N$. Also $M/P = \mathcal{O}^p(G/P) < G/P$ and $|G: M| = p$. By Lemma 4.2, there exists an irreducible constituent η of θ^P which is invariant in M. Now η is a p-character of P by Lemma 2.4 and thus there exists a unique extension $\hat{\eta}$ of η to M with $o(\eta) = o(\hat{\eta})$. We have either $\eta(1) = \theta(1)$ or $\eta(1) = p\theta(1)$. In the latter case, it is clear that $\hat{\eta}$ must be invariant in G and hence it is extendible to $\chi \in Irr(G)$. Now G/P does not have a normal S_p subgroup and thus has some irreducible character β of degree divisible by p. Since χ_P is irreducible, $\beta\chi \in Irr(G)$ and this contradicts $e(G, N, \theta) = 1$. Therefore we must have $\eta(1) = \theta(1)$.

We claim now that $e(G/N) = 1$. Let $\varphi \in Irr(M/N)$ with $p | \varphi(1)$. It suffices to show that $p^2 \nmid \varphi(1)$ and that φ is invariant in G. Now $\hat{\eta}\varphi \in Irr(M)$ and $p^2\theta(1) \nmid (\hat{\eta}\varphi)(1)$. Thus $p^2 \nmid \varphi(1)$. Also $(\hat{\eta}\varphi)^G$ is not irreducible so that $\hat{\eta}\varphi$ is invariant in G. Now let $x \in G$. Then $\hat{\eta}\varphi = (\hat{\eta}\varphi)^x = \hat{\eta}^x\varphi^x$. Since $\hat{\eta}^x$ and $\hat{\eta}$ are both extensions of θ to M, there exists a linear character λ of M/N with $\hat{\eta}^x = \lambda\hat{\eta}$. Substituting in the above, we obtain $\hat{\eta}\varphi = \lambda\hat{\eta}\varphi^x$. Since $\hat{\eta}$ is an extension of θ and φ and $\lambda\varphi^x$ are irreducible characters of H/N, it follows by Proposition 1.4 that $\varphi = \lambda\varphi^x$. Applying this to the complex conjugate character $\bar{\varphi}$, we obtain $\bar{\varphi} = \lambda\bar{\varphi}^x$, and thus $\varphi = \bar{\lambda}\varphi^x$. This yields $\lambda\varphi^x = \bar{\lambda}\varphi^x$ and $\lambda^2\varphi^x = \varphi^x$.

Now $o(\eta) = o(\hat{\eta}^x)$. We have $\det(\hat{\eta}^x) = \det(\lambda\hat{\eta}) = \lambda^f \det(\hat{\eta})$ where $f = \hat{\eta}(1)$ is a power of p. It follows that $o(\lambda)$ is a power of p, and since $p > 2$, λ is a power of λ^2. Since $\varphi^x = \lambda^2\varphi^x$, we obtain $\varphi^x = \lambda\varphi^x = \varphi$. Since $x \in G$ was arbitrary, φ is invariant in G and we have thus shown that $e(G/N) = 1$.

We may now assume without loss that $N = 1$. In the notation of [6], P has $r.x.1$ and by Theorem C of that paper, either P has an abelian subgroup of index p or else $|P: \mathbf{Z}(P)| = p^3$. It follows that either P has a characteristic abelian subgroup of index p or $|P: \mathbf{Z}(P)| \leqq p^3$. We claim that there exists $A \lhd G, A \subseteqq P$ with $|P: A| = p$ and A abelian. If this is not the case then $|P:\mathbf{Z}(P)| \leqq p^3$. Let S be an $S_{p'}$ subgroup of M. Then $U = [P, S] \lhd G$ since for $g \in G$, $S^g = S^x$ for some $x \in P$. We claim that $U \subseteqq \mathbf{Z}(P)$. Otherwise $V = U\mathbf{Z}(P) > \mathbf{Z}(P)$ and we choose $Y \lhd G$, maximal such that $\mathbf{Z}(P) \subseteqq Y < V$. Let $C/Y = \mathbf{C}_{G/Y}(V/Y)$. Then V/Y is a faithful irreducible G/C module. Now $|V/Y| \leqq p^3$ and $p \geqq 5$ and hence it follows from Proposition 4.3 that $p \nmid |G/C|$. Since $\mathcal{O}^{p'}(G) = G$, it follows that $G = C$ and thus $[V, G] \subseteqq Y$. In particular $[U, S] \subseteqq Y \cap U < U$. Since $U = [P, S] = [P, S, S]$, this is a contradiction and thus $U \subseteqq \mathbf{Z}(P)$. It follows that $\mathbf{Z}(P) \supseteqq P \cap \mathcal{O}^p(G)$.

Since P is not abelian, $P/\mathbf{Z}(P)$ is not cyclic and thus $G/\mathcal{O}^p(G)$ is not cyclic. It follows that there exists a subgroup $M_0 \lhd G$, with $M \neq M_0$ and $|G: M_0| = p$. Now $\mathcal{O}_p(M_0) = M_0 \cap P$ is not an S_p subgroup of M_0. By induction, $M_0 \cap P$ is abelian. Since $|P: M_0 \cap P| = p$ and $M_0 \cap P \lhd G$, the claim is established and A exists.

Suppose λ is a linear character of A which is not invariant in P. Let $T = \mathscr{I}_G(\lambda)$. Then, $P \cap T = A$ and hence $p \,|\, |G: T|$. By Corollary 1.3, it follows that $e(T, A, \lambda) = 0$ and $p^2 \nmid |G: T|$. Since λ is obviously a p-character, it follows from Corollary 3.4 that T/A has a normal S_p subgroup, of order exactly p. Let U be the group of linear characters of A. Then G/A acts on U and we let $Z = \mathbf{C}_U(P/A)$. The above argument shows that if $u \in U - Z$, then $\mathbf{C}_{G/A}(u)$ has a normal S_p subgroup of order p.

Let $P/A = \langle x \rangle$ and let $W = [U, x]$. Then the map $f: u \to [u, x]$ defines a homomorphism from U onto W and $\ker f = Z$. Set $Y = G/A$ and $P_0 = P/A \lhd Y$. Now $\mathbf{C}_Y(P_0)$ has index dividing $p - 1$. However

$\mathcal{O}^{p'}(Y) = Y$ and it follows that $P_0 \subseteq \mathbf{Z}(Y)$. Therefore, for $y \in Y$ and $u \in U$, we have $f(u^y) = f(u)^y$ and f is a homomorphism of Y-modules. Also, from $P_0 \subseteq \mathbf{Z}(Y)$, it follows that Y has a normal p-complement and thus so does every subgroup.

Since P is not abelian, $A \not\subseteq \mathbf{Z}(P)$ and it follows that x acts nontrivially on U. Therefore $W > 1$ and hence $V = W \cap Z > 1$. Now choose $w \in V$, $w \neq 1$. Let K be the normal p-complement of $\mathbf{C}_Y(w)$. Then K fixes the inverse image of w under f, which is a coset of Z. It follows (by Theorem 1 of [2] for instance), that K fixes some element $u \in U$ with $f(u) = w$. In particular, $u \notin Z$ so $\mathbf{C}_Y(u)$ has the normal S_p subgroup, P_1, of order p. Now K is a full p-complement for $\mathbf{C}_Y(u)$ since $\mathbf{C}_Y(u) \subseteq \mathbf{C}_Y(w)$. Hence $\mathbf{C}_Y(u) = K \times P_1$ and

$$\mathbf{C}_Y(w) = K \times P_1 \times P_0 .$$

Now, $\mathbf{C}_Y(V) \subseteq \mathbf{C}_Y(w)$ and thus has a normal S_p subgroup. Since $\mathcal{O}_p(Y) = P_0$, P_0 is a full S_p subgroup of $\mathbf{C}_Y(V)$.

Now suppose $v \in V$ with $P_1 \not\subseteq \mathbf{C}(v)$. Let P_2 be the subgroup of order p in $\mathbf{C}_Y(vu)$. Then $P_2 \neq P_1$ and $P_2 \neq P_0$. Furthermore, since $f(vu) = w$, $P_2 \subseteq \mathbf{C}_Y(vu) \subseteq \mathbf{C}_Y(w)$ and thus $P_2 \subseteq P_0P_1$. We may therefore choose $y \in P_1$ with $xy \in P_2$. Then $uv = (uv)^{xy} = u^{xy}v^y = u^xv^y$. However, $w = f(u) = u^{-1}u^x$ and $u^x = uw$. Hence $uv = uwv^y$ and $[y, v] = v^{-y}v = w$. Since $wy = yw$, it follows that $1 = [y^p, v] = w^p$ and w has order p. Since $w \in V$ was arbitrary, V is elementary abelian. Also from $[y, v] = w$, it follows that $[P_1, v] = \langle w \rangle$. Since $v \in V$ was arbitrary, not centralized by P_1, it follows that $[P_1, V] = \langle w \rangle$. Therefore $\mathbf{C}_V(P_1)$ has codimension 1 in V. Now choose $w^* \in \mathbf{C}_V(P_1)$ with $w^* \neq 1$. Repeating the above reasoning with w^* in place of w, we conclude that $[P_1^*, V] = \langle w^* \rangle$, where $P_1^* \times P_0$ is a normal S_p subgroup of $C_Y(w^*)$. By the choice of w^*, $P_1 \subseteq \mathbf{C}_Y(w^*)$ and thus $P_1 \subseteq P_1^* \times P_0$. Since $[P_0, V] = 1$, $\langle w \rangle = [P_1, V] \subseteq [P_1^*, V] = \langle w^* \rangle$. It follows that $\mathbf{C}_V(P_1) = \langle w \rangle$ and hence $|V| = p^2$. Given any basis $\{v, w\}$ for V, the above argument shows that there exists $y \in Y$ with $[y, v] = w$ and thus Y acts irreducibly on V. Since $p > 3$, Proposition 4.3 applies and $p \nmid |Y : \mathbf{C}_Y(V)|$. It follows that Y centralizes V which is a contradiction. The proof is complete.

COROLLARY 5.3 *If $p > 3$, then $u(1) = 2$.*

Proof. It suffices to show $u(1) \leq 2$. Let $e(G, N, \theta) = 1$ with θ a p-character and G/N solvable. If $\mathcal{O}_p(G/N) = H/N$ is an S_p subgroup of G/N, then by Lemma 1.6, d.l.$(H/N) \leq 2$ and nothing remains to be shown. Otherwise H/N is abelian. Choose an irreducible constituent ψ of θ^H which is invariant in U, where $U/H = \mathcal{O}_{p'}(G/N)$.

(Lemma 4.2). Let $T = \mathscr{I}_G(\psi)$. Then $\mathscr{O}_p(T/H) = 1$ and $e(T, H, \psi) \leqq 1$. If $e(T, H, \psi) = 0$ then since $v(0) = 0$, $p \nmid |T:H|$ and $p^2 \nmid |G:T|$ by Corollary 1.3. Thus $p^2 \nmid |G: H|$ and the result follows. If $e(T, H, \psi) = 1$ then $p \nmid |G: T|$ and the result follows from $v(1) = 1$.

6. The assumption $p > 3$ was used twice in the proof of Theorem 5.2. In this section we give examples to show that both uses were essential.

EXAMPLE 6.1. Let P be the group of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & x^3 \\ 0 & 0 & 1 \end{bmatrix} = M(x, y)$$

where $x, y \in GF(27)$. Then $|P| = 3^6$ and

$$P' = \mathbf{Z}(P) = \{M(0, y) | y \in GF(27)\} .$$

Let $\lambda \in GF(27)$ have order 13. Then the map $M(x, y) \to M(x\lambda, y\lambda^4)$ is an automorphism of P of order 13. Denote this automorphism by σ_λ and let M be the split extension $P \langle\sigma_\lambda\rangle$. Now $GF(27)$ has an automorphism τ of order 3 and we let τ act on M in the natural manner, with $(\sigma_\lambda)^\tau = \sigma_{\lambda^\tau}$. Let $G = M \langle\tau\rangle$. We claim that $e(G) = 1$, but $\mathscr{O}_3(G) = P$ is not abelian.

It suffices to check that every irreducible character of P is stabilized by some element of order 3 in G/P. Now τ fixes the two linear characters of P whose kernel is $[P, \tau]$. It is not hard to show that $P \langle\tau\rangle/[P', \tau]$ has center of index 3^3 so all of its irreducible nonlinear characters have degree 3. It follows that τ fixes all six nonlinear irreducible characters of P with kernel containing $[P', \tau]$. Since σ acts transitively on hyperplanes of P/P' and of P', it follows that every irreducible character of P is conjugate in M to a character fixed by τ and this proves the claim. Note that G contains no normal abelian subgroup A of index 3 in P. Also, d.l.$(P \langle\tau\rangle) = 3$.

EXAMPLE 6.2. Let $A = \langle x_1, x_2, y_1, y_2\rangle$ be elementary abelian of order 3^4. Let $Y = \langle\sigma\rangle \times S$ where σ has order 3 and $S \cong SL(2, 3)$. Let Y act on A so that S acts in its natural manner on $\langle x_1, x_2\rangle$ and on $\langle y_1, y_2\rangle$ with $x_1 \to y_1$ and $x_2 \to y_2$ defining an S-isomorphism. Let $x_i^\sigma = x_i y_i$ and $y_i^\sigma = y_i$. Let G be the split extension AY. Now $\mathscr{O}_3(G) = A \langle\sigma\rangle$ is not abelian.

To show that $e(G) = 1$, it suffices to show that every linear character of A is fixed by some element of Y of order 3. Let U be the group of linear characters of A and let $V \subseteqq U$ be those whose kernels contain $\langle y_1, y_2\rangle$. The unique element of order 2 of Y fixes no nonidentity element of U and hence for $1 \neq u \in U$, $C_Y(u)$ is a 3-group. Now the 3-subgroups of Y, either contain σ or else have order 3. Since $C_U(\sigma) = V$, it follows that if $u \in U - V$, then $|C_Y(u)| \leqq 3$.

Each subgroup of order 3 of Y must centralize a subgroup of order at least 9 in U since U is elementary abelian of order 3^4. Since $C_Y(V) = \langle\sigma\rangle$, it follows that each of the 12 subgroups of Y of order 3, different from $\langle\sigma\rangle$, centralize at least six elements of U-V. Since these sets are disjoint, this accounts for all 72 elements of $U - V$ and the result follows.

In example 6.2, even though the normal abelian subgroup A does exist, the conclusion of Theorem 5.2 does not hold. Therefore, the second assumption that $p > 3$ was essential. Note that Example 6.1 shows that $u(3) = 3$.

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