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STRUCTURE OF NOETHER LATTICES WITH JOIN-PRINCIPAL MAXIMAL ELEMENTS

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In this paper we explore the structure of Noether lattices with join-principal maximal elements.

Results which completely specify the structure of certain special classes of Noether lattices, and relate them to lattices of ideals of Noetherian rings, have been obtained in [1], [2], [3], [4], [7], and [8]. For example, in [7] we showed that if every maximal element of a Noether lattice $\mathscr{L}$ is meet-principal, then $\mathscr{L}$ is distributive and can be represented as the lattice of ideals of a Noetherian ring. Moreover, for distributive Noether lattices, the condition that every maximal element is meet-principal is equivalent to representability. In a more recent paper [8], we began considering the complementary case of a Noether lattice in which every maximal element is join-principal in order to determine the extent of the relationship between the two situations. There we showed that if 0 is prime in $\mathscr{L}$ (and every maximal element is join-principal), then $\mathscr{L}$ is distributive and representable. Hence, if 0 is prime, the assumptions that every maximal element is meet-principal and that every maximal element is join-principal are equivalent, and either implies representability.

In this paper, we continue the investigation begun in [8]. Our results extend the class of Noether lattices for which embedding and structure theorems are known, and also introduce a construction process for Noether lattices which leads to new examples.

In § 1, we show that in a local Noether lattice $(\mathscr{L}, M)$ in which M is join-principal and not a prime of 0, the maximal element M has a minimal base $E_1, \cdots, E_k$ of independent principal elements (i.e., $E_i \wedge (E_1 \vee \cdots \vee \hat{E}_i \vee \cdots \vee E_k) = 0$ for $i = 1, \cdots, k$). And we use this result to show that if M is join-principal and not a prime of 0, then $\mathscr{L}$ is distributive. In § 2, we obtain structure and embedding theorems for distributive local Noether lattices with join-principal maximal elements. In § 3, we investigate some of the consequences of our results outside of the local case.

We adopt the terminology of [5].

1. Let $(\mathscr{L}, M)$ be a local Noether lattice and let $B \in \mathscr{L}$. The quotient B/MB is a finite dimensional complemented modular lattice and the number of elements in any minimal set of principal elements with join B is the dimension of the quotient B/MB ([4], [6]). Hence

if $E_1, \cdots, E_s$ is any set of principal elements with the property that the elements $E_i \vee MB$ are independent in B/MB, then $E_1, \cdots, E_s$ can be extended to a minimal base for B. We will have occasion to use these observations in what follows.

In this section we show that if $(\mathscr{L}, M)$ is a local Noether lattice in which M is join-principal and not a prime of 0, then $\mathscr{L}$ is distributive.

We begin with a lemma.

LEMMA 1.1. *Let $(\mathscr{L}, M)$ be a local Noether lattice in which M is join-principal and not a prime of* 0. *Let $E_1, \cdots, E_k$ be a minimal base for M and, for each $i = 1, \cdots, k$, set $C_i = E_1 \vee \cdots \vee \hat{E}_i \vee \cdots \vee E_k$. Then each of the elements $C_i (i = 1, \cdots, k)$ is prime.*

Proof. Since M is principal in $\mathscr{L}/C_i$ $(i = 1, \cdots, k)$, each of the elements C_i is either prime or M-primary [7]. Assume that C_r is M-primary. And let n be the least positive integer such that $E_r^{n+1} \leqq C_r$. Then $E_r^{n+1} \leqq MC_r$. For, if not, there exist principal elements $F_1, \cdots, F_s$ among $E_1, \cdots, \hat{E}_r, \cdots, E_k$ such that $E_r^{n+1}, F_1, \cdots, F_s$ is a minimal base for C_r. But then $E_r, F_1, \cdots, F_s$ is a minimal base for $M = E_r \vee C_r$. Since C_r, by definition, has fewer elements in a minimal base than M, this is a contradiction. Hence $E_r^{n+1} \leqq MC_r$, as claimed. Consequently, $M^{n+1} \leqq MC_r$, and therefore

$$E_r^n \leqq M^n \vee (0 : M) = M^{n+1} : M = MC_r : M = C_r \vee (0 : M) = C_r ,$$

since M is join-principal and not a prime of 0. Since $E_r^n \not\leqq C_r$, this leads to a contradiction. Hence, each of the elements C_i is prime.

LEMMA 1.2. *Let $(\mathscr{L}, M)$ be a local Noether lattice in which M is join-principal and not a prime of* 0. *Then, in the notation of Lemma* 1.1, $C_1 \wedge \cdots \wedge C_k = 0$.

Proof. Let $E_1, \cdots, E_k$ and $C_1, \cdots, C_k$ be as in Lemma 1.1. We first show that for $1 \leqq r < s \leqq k$, $E_rE_s = 0$. Hence, suppose that $E_rE_s \neq 0$, and let n be a positive integer such that $E_rE_s \leqq M^n$ and $E_rE_s \not\leqq M^{n+1}$. Then E_rE_s can be used in a minimal base for M^n. Now, since M is join-principal and not a prime of 0, it follows from the relation $M^{nk+n} = M^{nk}(E_1^n \vee \cdots \vee E_k^n)$ that the elements $E_1^n, \cdots, E_k^n$ form a minimal base for M^n. Hence, for some i, $1 \leqq i \leqq k$, $M^n = E_rE_s \vee E_1^n \vee \cdots \vee \hat{E}_i^n \vee \cdots \vee E_k^n$. But then $M^n \leqq C_i$, which contradicts Lemma 1.1. It now follows that, for each s $(1 \leqq s \leqq k)$, $C_s \wedge E_s = (C_s : E_s)E_s = C_sE_s = 0$, since C_s is prime and $E_s \not\leqq C_s$. Hence by modularity $C_1 \wedge \cdots \wedge C_s = E_{s+1} \vee \cdots \vee E_k$ for $s \leqq k$, so that $C_1 \wedge \cdots \wedge C_k = 0$.

We are now in a position to establish the main result of the section.

THEOREM 1.3. *Let* ($\mathscr{L}$, M) *be a local Noether lattice in which* M *is a join-principal and not a prime of* 0. *Then* $\mathscr{L}$ *is distributive.*

Proof. Let $E_1, \cdots E_k$ and $C_1, \cdots, C_k$ be as in Lemma 1.1. A simple inductive argument using modularity proves that

$$(\vee E_i^{j(i)}) \wedge (\vee E_i^{k(i)}) = \vee E_i^{\max(j(i),k(i))}$$

with the convention that E_i^∞ means 0. Thus it suffices to show that the only principal elements in $\mathscr{L}$ are 0, I and the powers E_i^n of the elements $E_1, \cdots, E_k$. If $k = 1$, the result is immediate, so assume $k \geqq 2$. Let E be any principal element of $\mathscr{L}$ different from 0 and I. We assume that the elements $E_1, \cdots, E_k$ are arranged so that $E \leqq C_i$ for $i > r$ and $E \not\leqq C_i$ for $i \leqq r$. Set $C = C_1 \wedge \cdots \wedge C_r$ and consider $\mathscr{L}/C$. Since M is principal in each of the local Noether lattices $\mathscr{L}/C_i$ $(i = 1, \cdots, k)$, it follows by Lemmas 1.1 and 1.2 that the primes of $\mathscr{L}/C$ are just M and $C_1, \cdots, C_r$. Hence, by the choice of E, the element $E \vee C$ is M-primary in $\mathscr{L}/C$, and therefore also in $\mathscr{L}$. Let n be a positive integer such that $M^{n+1} \leqq E \vee C$ and $M^n \not\leqq E \vee C$, then, by modularity,

$$\begin{aligned} M^{n+1} \vee C &= C \vee ((M^{n+1} \vee C) \wedge E) \\ &= C \vee ((M^{n+1} \vee C) : E)E\,. \end{aligned}$$

Hence, either $M^{n+1} \leqq C \vee ME$ or $(M^{n+1} \vee C) : E = I$. In the first case, however,

$$M^n \leqq M^{n+1} : M \leqq (C \vee ME) : M = (C : M) \vee E = C \vee E\,,$$

which contradicts the choice of n. Hence $(M^{n+1} \vee C) : E = I$ and $E \leqq M^{n+1} \vee C$. Then $E \vee C = M^{n+1} \vee C$, so by the join-irreducibility of principal elements in a local Noether lattice, it follows that $E \vee C = E_1^{\varphi(1)} \cdots E_k^{\varphi(k)} \vee C$, for some nonnegative integers $\varphi(1), \cdots, \varphi(k)$. On the other hand, $E_i \leqq C$ for $i > r$ and $E \not\leqq C$, so $\varphi(i) = 0$ for $i > r$. Now, if $i \neq j$ and $1 \leqq j \leqq r$, then $E_i \vee C \leqq C_j$. It follows that $r \leqq 1$, and hence that $E \leqq C_2 \wedge \cdots \wedge C_k$. Then by the proof of Lemma 1.2, $C_2 \wedge \cdots \wedge C_k = E_1$ and $ME_1^n = E_1^{n+1}$, for all n. Hence, there exists a positive integer u such that $E \leqq E_1^u$ and $E \not\leqq ME_1^u = E_1^{u+1}$. Since E_1 is principal, it is now immediate that $E = E_1^u$.

We note that if ($\mathscr{L}$, M) is a local Noether lattice in which $M^2 = 0$, then M is join-principal. Since such a Noether lattice need not be distributive, the statement of Theorem 1.3 need not be valid without the assumption that M is not a prime of 0. On the other hand, if $\mathscr{L}$

is an arbitrary Noether lattice in which every maximal element is join-principal, then the number of maximal primes associated with 0 is finite. Hence, at most finitely many of the localizations $\mathscr{L}_M$ (M maximal) are nondistributive.

2. Let $(\mathscr{L}_1, M_1)$ and $(\mathscr{L}_2, M_2)$ be local Noether lattices, and let $\mathscr{L} = \{(A, B) \in \mathscr{L}_1 \oplus \mathscr{L}_2;\ A = I$ if and only if $B = I\}$. It is clear that $\mathscr{L}$ is a sub-multiplicative-lattice of $\mathscr{L}_1 \oplus \mathscr{L}_2$. Moreover, if E_1 and E_2 are principal elements of $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, with $E_1 \neq I$ and $E_2 \neq I$, then the elements $(E_1, 0)$ and $(0, E_2)$ are principal in $\mathscr{L}$. Hence $\mathscr{L}$ is a local Noether lattice with maximal element (M_1, M_2). We refer to $\mathscr{L}$ as the *local direct sum of* $\mathscr{L}_1$ *and* $\mathscr{L}_2$. An alternative characterization is given by $\mathscr{L} = (M_1 \mid 0 \oplus M_2 \mid 0) \cup \{(I, I)\}$.

In this section we continue our investigation of a local Noether lattice $(\mathscr{L}, M)$ with join-principal maximal element. However, we drop the hypothesis that M is not a prime of 0 and consider, instead, the general distributive case. Our main result is that a distributive local Noether lattice $(\mathscr{L}, M)$, in which M is join-principal, is the local direct sum of local Noether lattices with principal maximal elements. We begin with an extension of Lemma 1.2.

LEMMA 2.1. *Let $(\mathscr{L}, M)$ be a distributive local Noether lattice in which M is join-principal. Let $E_1, \cdots, E_k$ be a minimal base for M. Then $E_i \wedge E_j = 0$ for all $i \neq j$.*

Proof. For each $i = 1, \cdots, k$, set $C_i = E_1 \vee \cdots \vee \hat{E}_i \vee \cdots \vee E_k$. Then

$$M = M^2 : M = (MC_i \vee E_i^2) : M = C_i \vee (E_i^2 : M)$$

and

$$E_i \vee (E_i^2 : M) = (ME_i \vee E_i^2) : M = ME_i : M = E_i \vee (0 : M) ,$$

so because

$$(E_i^2 : M) = (E_i^2 : M) \wedge (E_i \vee 0 : M) = 0 : M \vee ((E_i^2 : M) \wedge E_i)$$

by modularity, we have that

$$M = C_i \vee (0 : M) \vee ((E_i^2 : M) \wedge E_i) = C_i \vee (0 : M) \vee (E_i^2 : ME_i)E_i ,$$

$i = 1, \cdots, k$. Since principal elements are join-irreducible in a local Noether lattice, since $\mathscr{L}$ is distributive, and since $E_i \nleqq C_i$, it follows that either $E_i \leqq 0 : M$ or $E_i \leqq (E_i^2 : ME_i)E_i$, $i = 1, \cdots, k$.

Assume that $E_r \leqq (E_r^2 : ME_r)E_r$. Then $E_r^2 : ME_r = I$, so $ME_r = E_r^2$. Hence $M = ME_r : E_r = E_r^2 : E_r = E_r \vee (0 : E_r)$. It follows that $E_i \leqq E_r \vee (0 : E_r)$ for all i, and that $E_i \leqq 0 : E_r$ for $i \neq r$ since $\mathscr{L}$ is distributive and E_i is join-irreducible. Therefore $C_iE_i = 0$ $(i = 1, \cdots, k)$.

Now, assume that $1 \leqq i < j \leqq k$ and let E be a principal element such that $E \leqq E_i \wedge E_j$. Suppose that $E \neq 0$ and choose integers u and v such that $E \leqq E_i^u \wedge E_j^v$, $E \nleqq E_i^{u+1}$ and $E \nleqq E_j^{v+1}$. Then $E \leqq E_i^u$ and $E \nleqq ME_i^u$, so $E = E_i^u$. Similarly $E = E_j^v$, so $E_i^u = E = E_j^v$. Then $u > 1$ and $v > 1$, so $ME_i^{u-1} = ME_j^{v-1}$. It follows that $E_i^{u-1} \vee (0:M) = E_j^{v-1} \vee (0:M)$, so that either $E_i^{u-1} \leqq E_j^{v-1}$ or $E_i^{u-1} \leqq 0:M$. In either case, $E_i^u = 0$. Hence $E = 0$ and $E_i \wedge E_j = 0$.

THEOREM 2.2. *Let $(\mathscr{L}, M)$ be a distributive local Noether lattice. Then M is join-principal if, and only if, $\mathscr{L}$ is the (finite) local direct sum of local Noether lattices with principal maximal elements.*

Proof. Assume that $(\mathscr{L}, M)$ is a distributive local Noether lattice in which M is join-principal. Let $E_1, \cdots, E_k$ be a minimal base for M. And for each $i = 1, \cdots, k$, let $(\mathscr{L}_i, M_i)$ be a local Noether lattice such that M_i is principal and $M_i^n = 0$ if, and only if, $E_i^n = 0$. Since $\mathscr{L}$ is distributive, it follows by Lemma 2.1 and [2] that every element $A \in \mathscr{L}$ has a unique minimal basis consisting of powers of the elements $E_1, \cdots, E_k$. If we set $E_i^\infty = 0$ and $E_i^0 = I$, then it is clear that the map $E_1^{n_1} \vee \cdots \vee E_k^{n_k} \rightarrow (M_1^{n_1}, \cdots, M_k^{n_k})$ is a multiplicative lattice isomorphism of $\mathscr{L}$ onto the local direct sum of $\mathscr{L}_1, \cdots, \mathscr{L}_k$.

The converse is clear.

COROLLARY 2.3. *Let $(\mathscr{L}, M)$ be a distributive local Noether lattice in which M is join-principal. Then $\mathscr{L}$ is Noether-lattice-embeddable in the lattice of ideals of a homomorphic image of a regular local ring.*

Proof. By Corollary 2.2, $\mathscr{L}$ is the local direct sum of local Noether lattices $(\mathscr{L}_1, M_1), \cdots, (\mathscr{L}_k, M_k)$, where, for each i, M_i is principal in $\mathscr{L}_i$. If M_i is nilpotent in $\mathscr{L}_i$, let n_i be the least positive integer such that $M_i^{n_i} = 0$; otherwise, let $n_i = \infty$. Let RL_k be the regular local Noether lattice introduced in [1], and let $X_1, \cdots, X_k$ be the minimal base for the maximal element of RL_k. Let A be the join of the elements X_iX_j and $X_i^{n_i}$ (where $X_i^\infty = 0$). Then $\mathscr{L}$ is clearly isomorphic to RL_k/A. Since RL_k is Noether-lattice-embeddable in the lattice of ideals of a regular local ring, [1], it follows that RL_k/A and $\mathscr{L}$ are embeddable in the lattice of ideals of a homomorphic image of a regular local ring.

3. In this section we interpret some of the implications of the results of §§1 and 2 outside of the local case.

We begin with a new characterization of the representable distributive Noether lattices.

THEOREM 3.1. *Let $\mathscr{L}$ be a Noether lattice. Then $\mathscr{L}$ is distributive and representable as the lattice of ideals of a Noetherian ring if, and only if, for each maximal element M of $\mathscr{L}$, M is join-principal and O_M is meet-irreducible.*

Proof. If $\mathscr{L}$ is distributive and representable, then each maximal element M is principal [7]. Consequently, $\mathscr{L}_M$ is a quotient of a regular local Noether lattice of altitude 1, and O_M is meet-irreducible.

Now, assume that $\mathscr{L}$ is a Noether lattice such that, for every maximal element M, M is join-principal and O_M is meet-irreducible. Fix M and consider $\mathscr{L}_M$. If $\{M\}$ is not a prime of 0 in $\mathscr{L}_M$, then by Lemma 2.1, O_M is meet-irreducible if, and only if, $\{M\}$ is principal. On the other hand, if $\{M\}$ is a prime of 0 in $\mathscr{L}_M$, then $\{M\}$ is the only prime of 0. In this case, let E be any principal element such that $E \leqq 0:\{M\}$. Then $\{M\}E = 0$, so E is a point in $\mathscr{L}_M$. Since the meet of any two points is 0 and O_M is irreducible by assumption, it follows that $0:\{M\}$ is itself a point and that $0:\{M\} \leqq A$, for every $A \neq 0$. Now, assume that $\{M\} \neq 0:\{M\}$, and let F be a principal element such that $F \leqq \{M\}$, $F \not\leq \{M\}^2$ and $\{M\}F \neq 0$. Then F is $\{M\}$-primary, so there is a nonnegative integer n such that $\{M\}^n \not\leqq F$ and $\{M\}^{n+1} \leqq F$. Hence $\{M\}^{n+1} = \{M\}^{n+1} \wedge F = (\{M\}^{n+1}:F)F$, and therefore either $\{M\}^{n+1}:F = I$ or $\{M\}^{n+1} \leqq \{M\}F$. In the first case, $\{M\}^{n+1} = F$, so $\{M\} = F$ by the choice of F. In the second case,

$$\{M\}^n \leqq \{M\}^{n+1}:\{M\} = \{M\}F:\{M\} = F \vee (0:\{M\}) = F\,,$$

a contradiction. Hence $\{M\}$ is principal in $\mathscr{L}_M$. It now follows by [7] that $\mathscr{L}$ is distributive and representable.

Recall that a Noether lattice $\mathscr{L}$ satisfies the *weak union condition* if, given elements A, B and C such that $A \not\leqq B$ and $A \not\leqq C$, it follows that there exists a principal element $E \leqq A$ such that $E \not\leqq B$ and $E \not\leqq C$. This concept was used in [7] to characterize the distributive Noether lattices which are representable. It is easy to see that if $\mathscr{L}$ is a Noether lattice which satisfies the weak union condition, then every localization $\mathscr{L}_M$ has the (weaker) property that, given primes $P_1, \cdots, P_k$ and an element A such that $A \not\leqq P_i$ $(i = 1, \cdots, k)$, there exists a principal element $E \leqq A$ such that $E \not\leqq P_i$ $(i = 1, \cdots, k)$. We say that a Noether lattice with this latter property satisfies the *union condition on primes*.

THEOREM 3.2. *Let $\mathscr{L}$ be a distributive Noether lattice such that, for every maximal element M, $\mathscr{L}_M$ satisfies the union condition on primes. If 0 has no embedded primes and if every maximal element is join-principal, then $\mathscr{L}$ is Noether-lattice-embeddable in the lattice of ideals of a Noetherian ring.*

Proof. Let $0 = Q_1 \wedge \cdots \wedge Q_k$ be a normal decomposition in which Q_i is P_i-primary. And let M be a maximal element of $\mathscr{L}$. If M is a prime of 0, then M is a minimal prime. On the other hand, by Lemma 1.1, if M is not a prime of 0, then 0 is prime in $\mathscr{L}_M$. Hence, if we assume that $P_1, \cdots, P_s$ are nonmaximal primes and that $P_{s+1}, \cdots, P_k$ are maximal primes, we have that

$$\mathscr{L} \cong \mathscr{L}/P_1 \oplus \cdots \oplus \mathscr{L}/P_s \oplus \mathscr{L}/Q_{s+1} \oplus \cdots \oplus \mathscr{L}/Q_k .$$

Then each of the summands $\mathscr{L}/P_i$ $(i = 1, \cdots, s)$ is isomorphic to the lattice of ideals of some Noetherian ring [8], and each of the summands $\mathscr{L}/Q_i$ $(i = s + 1, \cdots, k)$ is Noether-lattice-embeddable in the lattice of ideals of a Noetherian ring (Corollary 2.3). The conclusion is now immediate.

By the results of [9], it is easy to see that any Noether lattice of the type described in Theorem 3.2 has the property that every element has a unique normal decomposition. On the other hand, a Noether lattice with this latter property is the direct sum of local Noether lattices with nilpotent maximal elements and one-dimensional Noether lattices in which 0 is prime [9]. These observations lead to the following, the proof of which is similar to the proof of Theorem 3.2:

THEOREM 3.3. *Let $\mathscr{L}$ be a Noether lattice in which each maximal element is join-principal. Then the following are equivalent:*

(i) *Each element has a unique normal decomposition.*

(ii) *$\mathscr{L}$ satisfies the union condition on primes and 0 has no embedded primes.*

(iii) *$\mathscr{L}$ is the (finite) direct sum of Noether lattices with principal maximal elements and local Noether lattices with nilpotent maximal elements.*

If, in addition, $\mathscr{L}$ is distributive, then each of the above implies that $\mathscr{L}$ is Noether-lattice-embeddable in the lattice of ideals of a Noetherian ring.

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