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RELATIONS NOT DETERMINING THE STRUCTURE OF L

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A relation S is said to be determined up to isomorphism by relations R with respect to a theory K if for all models $\mathfrak{A}_1, \mathfrak{A}_2$ of $K, \mathfrak{A}_1$ restricted to R is isomorphic to $\mathfrak{A}_2$ restricted to R implies $\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_2$. In this paper simple necessary conditions for S to be determined up to isomorphism by R are given. These are applied in set theory to show there are (nonstandard) models of set theory with isomorphic ordinals and nonisomorphic constructible sets. The isomorphism on the ordinals may be taken to preserve many familiar arithmetic functions on the ordinals as addition, multiplication and exponentiation.

In this paper we show that the structure of the constructible sets of a model of set theory is not determined by the order-type of its ordinals, or, in fact, by its ordinals with various familiar arithmetic functions. This is shown by exhibiting (nonstandard) models of set theory with isomorphic ordinals and nonisomorphic constructible sets. The isomorphism on the ordinals may be taken to preserve familiar arithmetic functions.

These results are obtained by the use of certain simple general model-theoretic results developed in § 2. We define a relation S to be determined up to isomorphism by a set of relations R with respect to a theory K if $\langle A, R_A, S_A\rangle \models K$, $\langle B, R_B, S_B\rangle \models K$, $\langle A, R_A\rangle \approx \langle B, R_B\rangle$ implies $\langle A, R_A, S_A\rangle \approx \langle B, R_B, S_B\rangle$. We then give two simple sufficient conditions for S not to be determined up to isomorphism by R wrt K. Firstly, by a modification of a model-theoretic proof of Beth's theorem relating implicit and explicit definability, we show S is not determined up to isomorphism by R if there is a sentence σ such that the consequences about R of K, $K \cup \{\sigma\}$, $K \cup \{\neg \sigma\}$ are all the same. Using this, we show S is not determined up to isomorphism by R wrt K if there is a model $\mathfrak{A}$ of K in which the truth set of $\mathfrak{A}$ is not Turing-reducible to K join the truth set of $\mathfrak{A}$ restricted to R.

After illustrating simple applications of these results in § 2, we turn to the main set theory results in § 3. We observe that for any model $\mathfrak{A}$ of set theory, $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}\rangle \equiv \langle \omega^{\omega}, <\rangle$ which has recursive truth set [9] and that the truth set of $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, \tilde{\varepsilon}_{\mathfrak{A}}, \cong_{\mathfrak{A}}\rangle$ is not recursive (where F is the map defined up Gödel from $On \rightarrow L$, $\alpha \tilde{\varepsilon} \beta$ if $F(\alpha) \varepsilon F(\beta)$, $\alpha \cong \beta$ if $F(\alpha) = F(\beta)$ [8; 16]. Using this we may by our model-theoretic results immediately construct models of set theory with isomorphic ordinals and nonisomorphic constructible sets. This result is extended to include various functions on the ordinals by generalizing the above approach.

2. Notation and model theory results. Our notation will generally follow that suggested in [1] with the following additions and modifications: Capital italic letters will denote either relations and operations, or sets of relations and operations. Capital italic letters will also denote either relation and operation symbols, or sets of relations and operation symbols. Of course, we will use certain standard relation or operation symbols such as $<$, $+$, $\times$ etc.

Say A is a set of operation and relation symbols. Then $\Sigma(A)$ is the sentences of the first-order language with relation and operations symbols the set A. We let $Th_{\mathfrak{A}}$ be the set of all sentences true of $\mathfrak{A}$ in the object language of $\mathfrak{A}$. We let $Tr_{\mathfrak{A}}(A)$ be $Th_{\mathfrak{A}} \cap \Sigma(A)$. Finally $Cn(K) = \{\sigma \mid K \models \sigma\}$. For certain special subsets of $\Sigma(A)$ we use standard symbols, e.g., ZF for the axioms of Zermelo-Fraenkel set theory.

In general given an ω-enumeration of A, a set of relation and operation symbols, we use this to define godel numbers of elements of $\Sigma(A)$. We denote the godel numbers of sentences by the sentences themselves, and similarly the set of godel numbers of a set of sentences by the set of sentences.

Let K be a theory in $\Sigma(A, B)$ (where A, B are sets of relation and operation symbols). We say B is determined up to $\approx$ by A with respect to (wrt) K if for every models $\langle C_1, A_1, B_1\rangle, \langle C_2, A_2, B_2\rangle \models K$, $\langle C_1, A_1\rangle \approx \langle C_2, A_2\rangle$ implies $\langle C_1, A_1, B_1\rangle \approx \langle C_2, A_2, B_2\rangle$ (or equivalently if for every models $\langle C, A, B\rangle, \langle C, A, B'\rangle \models K$, $\langle C, A, B\rangle \approx \langle C, A, B'\rangle$). We say B is determined up to $\equiv$ by A with respect to K if for every models $\langle C_1, A_1, B_1\rangle, \langle C_2, A_2, B_2\rangle \models K$, $\langle C_1, A_1\rangle \approx \langle C_2, A_2\rangle$ implies $\langle C_1, A_1, B_1\rangle \equiv \langle C_2, A_2, B_2\rangle$. In general we omit mention of K when it is clear.

Trivially, B is not determined up to $\equiv$ by A implies B is not determined up to $\approx$ by A.

Let K be a theory in $\Sigma(A)$. We say K is complete wrt $\Sigma' \subseteqq \Sigma(A)$ if for every $\sigma \in \Sigma'$, $K \models \sigma$ or $K \models \neg \sigma$.

THEOREM 1. *The following* 4 *are equivalent*

(a) *For every model $\mathfrak{A}$ of K, $K \cup Tr_{\mathfrak{A}}(A)$ is complete wrt $\Sigma(A, B)$.*

(b) *For every $\sigma \in \Sigma(A, B)$, there is a $\tau \in \Sigma(A)$ such that*

$$K \models (\sigma \leftrightarrow \tau) .$$

(c) *B is determined up to $\equiv$ by A wrt K.*

(d) *For all models* $\langle C_1, A_1, B_1\rangle$ *and* $\langle C_2, A_2, B_2\rangle$ *of* K,

$$\langle C_1, A_1\rangle \equiv \langle C_2, A_2\rangle \rightarrow \langle C_1, A_1, B_1\rangle \equiv \langle C_2, A_2, B_2\rangle .$$

Proof. We show (b) $\Rightarrow$ (a) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b). The first three implications are trivial and left to the reader. (c) $\Rightarrow$ (b) is a routine application of the Robinson Joint Consistency Theorem [11]. Assume $(\exists\sigma\varepsilon\Sigma(A, B))(\forall\tau\varepsilon\Sigma(A))(\neg K \models (\sigma \leftrightarrow \tau))$. Let σ_0 be such a σ. So $K \cup \{\sigma_0\}$, $K \cup \{\neg\sigma_0\}$ are consistent and $\neg(\exists\tau\varepsilon\Sigma(A))(\tau\varepsilon Cn(K \cup \{\sigma_0\})$ and $(\neg\tau)\varepsilon Cn(K \cup \{\neg\sigma_0\}))$.

Relabel the symbols of B preserving arity so as to be symbols not in $A \cup B$. Call this new set B'. Let φ' in general be φ with symbols in B replaced by the corresponding ones in B'. So

$$\neg(\exists\tau\varepsilon\Sigma(A))(\tau\varepsilon Cn(K \cup \{\sigma_0\}) \text{ and } (\neg\tau)\varepsilon Cn(K' \cup \{\neg\sigma_0'\})),$$

i.e., the hypotheses of the joint consistency theorem. So there is a $\mathfrak{C} = \langle C, A, B, B'\rangle$ such that $\mathfrak{C} \models K \cup \{\sigma_0\} \cup K' \cup \{\neg\sigma_0'\}$. So $\langle C, A, B\rangle \models K \cup \{\sigma_0\}$, $\langle C, A, B'\rangle \models K \cup \{\neg\sigma_0\}$. So

$$\langle C, A, B\rangle \not\equiv \langle C, A, B'\rangle ,$$

i.e., B is not determined up to $\equiv$ by A.

Let K be a theory in $\Sigma(A, B)$. We say B is Turing determined by A wrt K if for every model $\mathfrak{A}$ of K, $Tr_{\mathfrak{A}}(A, B) \leqq_T K$ join $Tr_{\mathfrak{A}}(A)$. Note—we are here assuming ω-enumerations of A, B have been given and that the derived godel numbers of sentences are abusively denoted by the sentences themselves. Also if K is recursive, then this is equivalent to $Tr_{\mathfrak{A}}(A, B) \leqq_T Tr_{\mathfrak{A}}(A)$. ($\leqq_T$ is Turing reducible.)

THEOREM 2. *B is determined up to $\equiv$ by A* wrt K *implies B is Turing determined by A* wrt K.

Proof. By Theorem 1, if B is determined up to $\equiv$ by A, then $K \cup Tr_{\mathfrak{A}}(A)$ is complete wrt $\Sigma(A, B)$ for every model $\mathfrak{A}$ of K. But hence $Tr_{\mathfrak{A}}(A, B) \leqq_T K$ join $Tr_{\mathfrak{A}}(A)$.

REMARK. We see, thus, that B Turing determined by A wrt $K \rightarrow B$ determined up to $\equiv$ by A wrt $K \rightarrow B$ determined up to $\approx$ by A wrt $K \rightarrow B$ determined up to $=$ by A wrt K (i.e., B is implicitly definable in terms of A wrt K). The converses of these do not hold. We give examples

(1) Let F, G be unary relation symbols. Let $K \subseteqq \Sigma(F, G)$ be

$\{\exists!\, xF(x), \exists! xG(x), \neg(F(x) \wedge G(x))\}$. Then G is not determined up to $=$ by F wrt K. But G is determined up to $\approx$ by F wrt K.

(2) Let $<$ be a binary relation symbol, F be a unary relation symbol. Let $K \subseteqq \Sigma(<, F)$ be $Th\langle Q, <, 0\rangle$.

Then F is not determined up to $\approx$ by $<$ wrt K.

$$\text{(e.g., } \langle (Q \cap (-\infty, 0)) \cup (R \cap [0, \infty)), <, -1\rangle \not\approx \langle (Q \cap (-\infty, 0)) \cup (R \cap [0, \infty)), <, 1\rangle \text{ .)}$$

But F is determined up to $\equiv$ by $<$ wrt K (as K complete).

(3) Let $<$ be a binary relation symbol, F a unary relation symbol. Let $K \subseteqq \Sigma(<, F)$ be theory of dense linear order without endpoints $\cup$ $\{\exists x \exists y(F(x) \wedge F(y) \wedge (\forall z)(F(z) \rightarrow x = z \vee y = z))\}$. Then F is not determined up to $\equiv$ by $<$ wrt K (as $\exists!\, xF(x)$ is undecided). Yet in any model $\mathfrak{A}$: $Tr_{\mathfrak{A}}(<, F)$ is recursive and hence $\leqq_T Tr_{\mathfrak{A}}(<)$. So F is Turing determined by $<$ wrt K.

Remark. We give several simple well-known examples illustrating the use of Theorem 2.

(1) Let $K = \mathrm{ENT}$ (elementary number theory). Then $\times$ is not determined up to $\equiv$ by $<, +$.

Proof. Let $\mathfrak{N} = \langle \omega, <, +, \times\rangle$. So $T_1 = Tr_{\mathfrak{N}}(<, +)$ is recursive, $T_2 = Th_{\mathfrak{N}}$ is not. So $T_2 \not\leqq_T T_1$. So by Theorem 2, $\times$ is not determined up to $\equiv$ by $<, +$. In fact, this result also holds for $K =$ any other arithmetic set of formulae true about $\mathfrak{N}$. Similar results hold for arithmetic theories with $\mathfrak{Q} = \langle Q, <, +, \times\rangle$ as a model.

(2) Add to the standard symbols $(<, +, \times, 0, 1)$ of real closed fields an additional unary relation symbol i which is intended just to apply to integers.

Let $K = \mathrm{RCF}$ (the theory of real closed fields)

$$\cup \{i(0), i(x) \rightarrow i(x+1) \wedge i(x-1), \neg(\exists x)(0 < x < 1 \wedge i(x)), \neg(\exists x)(\forall y)(x \leqq y \leqq x + 1 \rightarrow \neg i(y))\}$$

$\cup$ any arithmetic set of formulae true about $\mathfrak{N}$ relativized to

$$x \geqq 0 \wedge i(x) .$$

Then i is not determined up to $\approx$ by $<, +, \times, 0, 1$.

Proof. As if $\mathfrak{R} = \langle R, <, +, \times, i\rangle$, then $Tr_{\mathfrak{R}}(<, +, \times)$ join K is arithmetic, but $Th_{\mathfrak{R}}$ is not and hence not $\leqq_T Tr_{\mathfrak{R}}(<, +, \times)$ join K.

Similar results are obtained for algebraically closed fields.

3. Applications to set theory. We show in this chapter that the structure of the constructible sets is not determined up to isomorphism by the order type of the ordinals, nor in fact, by certain fairly large classes of functions on the ordinals. We proceed as follows:

Let F be the map of On (the ordinals) onto L (the constructible sets) defined by Godel [8]. Let

$$\tilde{\varepsilon}(\alpha, \beta) =_{df} F(\alpha)\varepsilon F(\beta);\ \cong (\alpha, \beta) =_{df} F(\alpha) = F(\beta)\ .$$

Takeuti defines [16] a theory of primitive recursive functions and relations on the ordinals, which we will call ONT (Ordinal number theory). In particular he shows:

LEMMA 1. (a) *$\tilde{\varepsilon}, \cong$ are primitive recursive relations*

(b) *if $\langle On, <, PR\rangle \models \mathrm{ONT}$, then $\langle On, \tilde{\varepsilon}, \cong\rangle \models ZF,\ V = L$ and $\langle On, <, PR\rangle \approx \langle On_{\langle On,\tilde{\varepsilon},\cong\rangle}, <_{\langle On,\tilde{\varepsilon},\cong\rangle}, PR_{\langle On,\tilde{\varepsilon},\cong\rangle}\rangle$*

(c) *if $\mathfrak{A} \models ZF$, then $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}}\rangle \models \mathrm{ONT}$ (if $\mathfrak{A} \models ZF$, then $On_{\mathfrak{A}}$ are its ordinals, $<_{\mathfrak{A}}$ the $<$ relation on its ordinals and $PR_{\mathfrak{A}}$ the primitive recursive functions on the ordinals.)*

Furthermore $\cong$ is definable in terms of $\tilde{\varepsilon}$.

Proofs. Omitted.

So if $\tilde{\varepsilon}$ is not determined up to $\approx$ (or $\equiv$) by $<$ (i.e., if there are models $\langle On, <, PR_1\rangle, \langle On, <, PR_2\rangle \models \mathrm{ONT}$ such that $\langle On, <, \tilde{\varepsilon}_1\rangle \not\approx$ (or $\not\equiv$) $\langle On, <, \tilde{\varepsilon}_2\rangle$); then by Lemma 1b $\mathfrak{A}_i = \langle On, \tilde{\varepsilon}_i, \cong_i\rangle \models ZF,\ V = L$. Furthermore by Lemma 1b, $\langle On_{\mathfrak{A}_i}, <_{\mathfrak{A}_i}\rangle \approx \langle On, <\rangle$ and hence

$$\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}\rangle \approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}\rangle\ .$$

But by assumption $\langle On, \tilde{\varepsilon}_1\rangle \not\approx$ (or $\not\equiv$) $\langle On, \tilde{\varepsilon}_2\rangle$, and so $\mathfrak{A}_1 \not\approx$ (or $\not\equiv$) $\mathfrak{A}_2$. As $\mathfrak{A}_i \models V = L$, $\mathfrak{A}_i = \langle L_{\mathfrak{A}_i}, \varepsilon_{\mathfrak{A}_i}\rangle$. So there are models $\mathfrak{A}_1, \mathfrak{A}_2 \models ZF$ such that $\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}\rangle \approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}\rangle$ and $\langle L_{\mathfrak{A}_1}, \varepsilon_{\mathfrak{A}_1}\rangle \not\approx$ (or $\not\equiv$) $\langle L_{\mathfrak{A}_2}, \varepsilon_{\mathfrak{A}_2}\rangle$ if $\tilde{\varepsilon}$ is not determined up to $\approx$ (or $\equiv$) by $<$.

We usually will show $\tilde{\varepsilon}$ is not determined up to $\equiv$ by $<$ by showing it is, in fact, not Turing determined by $<$. Similarly and more generally we have

PROPOSITION 2. *If $\tilde{\varepsilon}$ is not Turing determined by $<$ and some class of primitive recursive functions, then the structure of L is not determined up to $\equiv$ by the structure of the ordinals with that class of primitive recursive functions on them.*

Let $\mathfrak{A}$ be any arbitrary model of ZF, $V = L$. Then as is well known $Th_{\mathfrak{A}}$ is not recursive. As $\mathfrak{A} \models V = L$,

$$\mathfrak{A} \approx \langle L_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}} \rangle \approx \langle On_{\mathfrak{A}}, \tilde{\varepsilon}_{\mathfrak{A}}, \cong_{\mathfrak{A}} \rangle .$$

So $Th_{\mathfrak{A}} \leqq_T Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, \tilde{\varepsilon})$. So $Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, \tilde{\varepsilon})$ is not recursive.

THEOREM 3. *$Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<)$ is recursive.*

Proof. Omitted. We recommend the interested reader examine the proof in [7].
Hence $Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, \tilde{\varepsilon}) \nleqq_T Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<)$. So $\tilde{\varepsilon}$ is not Turing determined by $<$. So by the above arguments we have

COROLLARY 4. *If ZF is consistent, then there are models $\mathfrak{A}, \mathfrak{A}' \models ZF$ with $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}} \rangle \sim \langle On_{\mathfrak{A}'}, <_{\mathfrak{A}'} \rangle$ but $\langle L_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}} \rangle \not\equiv \langle L_{\mathfrak{A}'}, \varepsilon_{\mathfrak{A}'} \rangle$.*

To extend this result to preserve various functions on the ordinals in the isomorphism, we need results similar to Theorem 3 in order to conclude $\neg(Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, A, \tilde{\varepsilon}) \leqq_T Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, A))$ for appropriate classes A of primitive recursive functions. Also we will wish to consider nonprimitive-recursive functions in A. So we will have to expand $PR_{\mathfrak{A}}$, i.e., we will have to change it to $\mathrm{ARITH}_{\mathfrak{A}}$, the arithmetic functions on the ordinals of $\mathfrak{A}$ which we will shortly define.

Theorem 3 will be extended by application of

LEMMA 5. *Let A be a set of primitive recursive functions. If there is a model $\mathfrak{A}$ of ZF and an ordinal $\alpha_{\mathfrak{A}} \varepsilon On_{\mathfrak{A}}$ such that*
1. $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}} \rangle \equiv \langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}} \rangle$ and
2. $Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, A, \tilde{\varepsilon}) \nleqq_T Tr_{\langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}} \rangle}(<, A)$.

then the structure of L is not determined up to $\equiv$ by the structure of the ordinals with the class A of primitive recursive functions on them.

Proof. For by 1, 2 above we have $\tilde{\varepsilon}$ is not Turing determined by A. So by Proposition 2 we are done.

We will show results of the form $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}} \rangle \equiv \langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}} \rangle$ by use of a technique of Ehrenfeucht [7]. Given models $\mathfrak{A}_1, \mathfrak{A}_2$ he defines 2 person finite games $G_n(\mathfrak{A}_1, \mathfrak{A}_2)$, $H_n(\mathfrak{A}_1, \mathfrak{A}_2)$ and shows that if for all n the second player has a winning strategy in $G_n(\mathfrak{A}_1, \mathfrak{A}_2)$ then $\mathfrak{A}_1 \equiv \mathfrak{A}_2$. A similar slightly stronger result holds for $H_n(\mathfrak{A}_1, \mathfrak{A}_2)$. So to generalize Corollary 4 we must merely by the above argument meet the hypothesis of the following lemma:

LEMMA 6. *Let A be a set of primitive recursive functions. If there is a model $\mathfrak{A}$ of ZF and an ordinal $\alpha_{\mathfrak{A}} \varepsilon On_{\mathfrak{A}}$ such that*
1. *Player* II *has a winning strategy in*

$$H_n(\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle, \langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle), \forall n$$

2. *As in Lemma* 5,

then the same conclusions as in Lemma 5 *hold.*

In particular, Ehrenfeucht has shown by these techniques that

THEOREM 7.

$$\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, +_{\mathfrak{A}}, x_{\mathfrak{A}}\rangle \equiv \langle \omega_{\mathfrak{A}}^{\omega^{\omega^{\omega}}}, <_{\mathfrak{A}}, +_{\mathfrak{A}}, \times_{\mathfrak{A}}\rangle .$$

Proof. [7].

Let $\mathfrak{A}$ be an arbitrary ω-model of ZF, i.e., an arbitrary model of ZF such that $\omega_{\mathfrak{A}} = \omega$. One can readily show that

$$\langle \omega_{\mathfrak{A}}^{\omega^{\omega^{\omega}}}, <_{\mathfrak{A}}, +_{\mathfrak{A}}, x_{\mathfrak{A}}\rangle \approx \langle \omega^{\omega^{\omega^{\omega}}}, <, +, \times\rangle$$

which has a Δ_1^1-truth set. On the other hand,

$$Th_{\mathfrak{A}} \leqq_T Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}}\rangle}(<, +, \times, \tilde{\varepsilon})$$

and $Th_{\mathfrak{A}}$ is not Δ_1^1 (for any ω-model of ZF includes a definable ω-model of analysis whose truth set is well-known to be not Δ_1^1).

So $Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}}\rangle}(<, +, \times, \tilde{\varepsilon}) \not\leqq_T Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, PR_{\mathfrak{A}}\rangle}(<, +, \times)$.

COROLLARY 8. *If ZF has ω-models there are models $\mathfrak{A}_1, \mathfrak{A}_2 \models ZF$ with $\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}, +_{\mathfrak{A}_1}, \times_{\mathfrak{A}_1}\rangle \approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}, +_{\mathfrak{A}_2}, \times_{\mathfrak{A}_2}\rangle$ but*

$$\langle L_{\mathfrak{A}_1}, \varepsilon_{\mathfrak{A}_1}\rangle \not\approx \langle L_{\mathfrak{A}_2}, \varepsilon_{\mathfrak{A}_2}\rangle .$$

Further extensions of these results involve the use of non primitive recursive functions. So we now define arithmetic functions and give certain elementary properties of them that are needed. A more detailed exposition, including the proofs of the elementary properties is presented in our dissertation [14].

A predicate $A(a_1 \cdots a_n)$ of ONT is called *arithmetic* if and only if there is a primitive recursive predicate $B(a_1 \cdots a_n x_1 \cdots x_m)$ and quantifiers $Q_1 \cdots Q_m$ such that

$$\vdash_{\overline{ONT}}(A(a_1 \cdots a_n) \leftrightarrow (Q_1 x_1) \cdots (Q_m x_m) B(a_1 \cdots a_n x_1 \cdots x_m)) .$$

The constant, relation and function symbols of ONT′ (theory of arithmetic functions on the ordinals) are those of ONT and the arithmetic function symbols described below. For each arithmetic predicate

$A(a_1 \cdots a_{n+1})$ such that $\vdash_{ONT'} (\exists! \, a_{n+1}) A(a_1 \cdots a_{n+1})$ we introduce a function symbol f_A whose intended meaning is to be the function defined by $f_A(a_1 \cdots a_n) = a_{n+1}$ if and only if $A(a_1 \cdots a_{n+1})$.

The axioms of ONT′ are those of ONT allowing, however, in all schema arithmetic function symbols together with the "definitions" of the arithmetic function symbols, i.e., if f_A is an arithmetic function such that $\vdash_{ONT'} (\exists! \, a_{n+1}) A(a_1 \cdots a_{n+1})$, then

$$f_A(a_1 \cdots a_n) = a_{n+1} \leftrightarrow A(a_1 \cdots a_{n+1})$$

is an axiom of ONT′.

A predicate $A(a_1 \cdots a_n)$ of ONT′ is called *arithmetic* if there is an arithmetic predicate $B(a_1 \cdots a_n)$ of ONT such that

$$\vdash_{ONT'} (A(a_1 \cdots a_n) \leftrightarrow B(a_1 \cdots a_n)) \, .$$

A predicate $A(a_1 \cdots a_{n+1})$ of ONT′ is called a *function* if

$$\vdash_{ONT'} (\exists! \, a_{n+1}) A(a_1 \cdots a_{n+1}) \, .$$

It is called an *arithmetic function* if

$$\vdash_{ONT'} A(a_1 \cdots a_{n+1}) \leftrightarrow f(a_1 \cdots a_n) = a_{n+1}$$

for some arithmetic function symbol f.

One can readily show using results and techniques of Takeuti for primitive recursive functions.

LEMMA 9a. *Every primitive recursive function is arithmetic.*

LEMMA 9b. *Every predicate of* ONT′ *is arithmetic, i.e., every predicate consisting only of arithmetic function symbols, constant, relation and function symbols of* ONT′, $=$, $\neg$, $\wedge$, $\vee$, $\exists$, $\forall$ *is arithmetic.*

LEMMA 9c. *If* $A_1(a), \cdots, A_n(a)$ *are arithmetic predicates such that* $\vdash_{ONT'} A_1(a) \vee \cdots \vee A_n(a)$ *and* $\vdash_{ONT'} \neg(A_i(a) \wedge A_j(a))$ *for* $i \neq j$ *and if* $f_1(a) \cdots f_n(a)$ *are arithmetic functions, then there exists an arithmetic function* $f(a)$ *such that*

$$\vdash_{ONT'} \bigwedge_{i \leq n} (A_i(a) \rightarrow f(a) = f_i(a)) \, .$$

As the arithmetic functions are definable in terms of the primitive recursive predicates and in terms of the primitive recursive functions we have that the results in Lemma 1b, 1c of Takeuti continue to hold, i.e., LEMMA 9d. *If* $\langle On, <, Arith\rangle \models$ ONT$'$, *then* $\mathfrak{A} = \langle On, \tilde{\varepsilon}, \cong\rangle \models ZF$, $V = L$ *and* $\langle On, <, Arith\rangle \approx \langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, Arith_{\mathfrak{A}}\rangle$.

LEMMA 9e. *If* $\mathfrak{A} \models ZF$, *then* $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, Arith_{\mathfrak{A}}\rangle \models$ ONT$'$.
Hence as with ONT we may conclude by use of ONT$'$ that

LEMMA 10. *Let A be a set of arithmetic functions.*
If there is a model $\mathfrak{A}$ of ZF and an ordinal $\alpha_{\mathfrak{A}} \varepsilon On_{\mathfrak{A}}$ such that

(1), (2) *as in Lemma* 5

then the structure of L is not determined up to $\equiv$ by the structure of the ordinals with the class A of arithmetic functions on it.

The above completes the necessary treatment of arithmetic functions. In the following extensions of Corollary 4, we will encounter cases where although $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle \equiv \langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle$ for an appropriate ordinal $\alpha_{\mathfrak{A}}$, it is not the case that $Th_{\langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, A_{\mathfrak{A}}\rangle}$ is Δ^1_1 or another similarly classified set which we can show $Th_{\mathfrak{A}}$ is not. Hence to handle this situation we put restrictions on the model $\mathfrak{A}$ in the hypothesis of Lemma 10; we require $\mathfrak{A}$ to be a nameable standard model of ZF [10]. (A nameable model is one all of whose sets are definable. If $\widetilde{ZF}$ is a "nice", e.g., finite or recursive, extension of ZF, then the minimal standard model of $\widetilde{ZF}$ is nameable.)

LEMMA 11. *Let A be a set of arithmetic functions.*
If there is a nameable standard model $\mathfrak{A}$ of ZF and an ordinal $\alpha_{\mathfrak{A}} \in On_{\mathfrak{A}}$ such that
1. *as in Lemma* 10
then the same conclusion as in Lemma 10 *holds.*

Proof. As $\mathfrak{A}$ is nameable, $Th_{\mathfrak{A}} \notin \mathfrak{A}$ (or else it would be definable contradicting the Tarski theorem on the definability of truth predicates [11]).

$$Th_{\mathfrak{A}} = Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, Arith_{\mathfrak{A}}\rangle}(\tilde{\varepsilon}) \leqq_T \quad Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, Arith_{\mathfrak{A}}\rangle}(<, \tilde{\varepsilon}) \,.$$

So as standard models are closed under $\leqq_T$, $Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, Arith_{\mathfrak{A}}\rangle}(<, \tilde{\varepsilon}) \notin \mathfrak{A}$. Hence this last set is $\not\leqq_T Tr_{\langle \alpha_{\mathfrak{A}}, <_{\mathfrak{A}}, Arith_{\mathfrak{A}}\rangle}(<, A)$, i.e., condition 2 of Lemma 10 is met.

The restriction in Lemma 11 to nameable models will often be unessential as we see in Lemma 12 and its applications.

LEMMA 12. *Let A be a set of arithmetic functions.*

If there is a formula F with 1 *free variable such that*

(1) *there is a standard model* $\mathfrak{A}$ *of* $ZF, (\exists!\, x)(F(x) \wedge Ord(x))$, $V = L$ *(call this* $\widetilde{ZF}$*)*

(2) $\forall n, \vdash_{\widetilde{ZF}}$ *Player* II *has a winning strategy in*

$$H_n(\langle On, <, A\rangle, \langle \alpha, <, A\rangle)$$

for $\alpha \in On$ *such that* $F(\alpha)$*, then the structure of* L *is not determined up to* $\equiv$ *by the structure of the ordinals with the class* A *of arithmetic functions on it.*

Proof. As $\widetilde{ZF}$ has a standard model, it has a nameable model $\mathfrak{B}$. [10].

So $\mathfrak{B} \models (\exists!\, x)(F(x) \wedge Ord(x))$. Let $\beta \varepsilon On_{\mathfrak{B}}$ such that $\mathfrak{B} \models F[\beta]$. By assumption 2, $\mathfrak{B} \models$ Player II has a winning strategy in

$$H_n(\langle On, <, A\rangle, \langle \beta, <, A\rangle) .$$

So condition 1 of Lemma 11 is true in $\mathfrak{B}$.

Remark. This lemma also holds for finite extensions of ZF.

Doner and Tarski (Doner and Tarski 67) define higher exponential functions O_γ as follows:

(1) $\alpha O_\gamma \beta = \alpha + \beta$ if $\gamma = 0$.

(2) $\alpha O_\gamma \beta = \lim_{\eta < \beta, \zeta < \gamma} ((\alpha O_\gamma \eta) O_\zeta \alpha)$ if $\gamma \geqq 1$.

Let $O(\alpha, \beta, \gamma) = \alpha O_\gamma \beta$

$$O^\delta(\alpha, \beta, \gamma) = \begin{cases} O(\alpha, \beta, \gamma) \text{ if } \gamma \leqq \delta \\ 0 \text{ otherwise} . \end{cases}$$

Lemma 13. *$O(\alpha, \beta, \gamma)$ is arithmetic.*

Proof. We, in fact, illustrate a general method of proving that the arithmetic functions are closed under recursion. The approach is to realize that given $\tilde{\varepsilon}$, $\cong$ we have a model of ZF in $\langle On, Arith\rangle$ in which we can by suitable means define the higher exponential functions on ordinals of this model of set theory. Then by means of the isomorphism of

$$\langle On_{\langle On, \tilde{\varepsilon}, =\rangle}, \tilde{\varepsilon}, Arith_{\langle On, \tilde{\varepsilon}, =\rangle}\rangle \xrightarrow{\approx} \langle On, <, Arith\rangle$$

we can convert these into the functions on On.

First we give several predicates needed in the discussion of $\langle On, \tilde{\varepsilon}, \cong\rangle$ and the map from $On_{\langle On, \tilde{\varepsilon}, =\rangle}$ to On. The reader may readily confirm that these are all arithmetic (usually by the use of some of the previous ones):

$M(x) =_{df} \neg(\exists\alpha')_{<\alpha}(\alpha' \cong \alpha)$. ($\alpha$ is a set of $\langle On, \tilde{\varepsilon}, \cong\rangle$. We, thus, successfully isolate a singll representative for each member of the model $\langle On, \tilde{\varepsilon}, \cong\rangle$ of ZF. We let M denote $\{x\varepsilon On \mid M(x)\}$. In particular $\cong \cap M \times M$ is $=$).

$\varepsilon^* =_{df} \in \cap M \times M$, $=^* =_{df} \cong \cap M \times M$ ($=$ ordinary $=$).

$UP(x) =_{df} x$ is an unordered pair in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$x = \{y, z\}^* =_{df} x = \{y, z\}$ in sense of $M, \varepsilon^*, =^*\rangle$.

$OP_2(x) =_{df} x$ is an ordered pair in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$x = \langle y, z\rangle^* =_{df} x = \langle y, z\rangle$ in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$OP_n(x) =_{df} x$ is an ordered n-tuple in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$x = \langle x_1 \cdots x_n\rangle^* =_{df} x = \langle x_1 \cdots x_n\rangle$ in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$Rel_n(x) =_{df} x$ is a n-ary relation in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$Fcn_n(x) =_{df} x$ is a n-ary function in sense of $\langle M, \varepsilon^*, =^*\rangle$.

$x = f(y_1 \cdots y_n)^* =_{df} x = f(y_1 \cdots y_n)$ in sense of $\langle M, \varepsilon^*, =^*\rangle$.

The following is primitive recursive

$$\begin{aligned} Ord(x) &=_{df} M(x) \wedge (\forall y)_{<x}(\forall z)_{<x}(y\varepsilon^* x \wedge z\varepsilon^* x \\ &\rightarrow (y\varepsilon^* z \vee z\varepsilon^* y \vee z = y)) \wedge (\forall y)_{<x}(\forall z)_{<x} \\ &(y\varepsilon^* z \vee z\varepsilon^* y \rightarrow z\varepsilon^* x)\ , \end{aligned}$$

i.e., $Ord(x)$ if and only if x is an ordinal of $\langle M, \varepsilon^*, =^*\rangle$ in sense of $\langle M, \varepsilon^*, =^*\rangle$. As a result by a standard application of primitive recursion $h(x)$ defined on $\langle M, \varepsilon^*, =^*\rangle$ ordinals as follows is primitive recursive:

$$h(x) = (\mu\alpha)_{<x'}((\forall y)(y^*\varepsilon x \rightarrow h(y) < \alpha))\ .$$

That is h is the map of $\langle M, \varepsilon^*, =^*\rangle$-ordinals isomorphically to On.

Hence as $+$ on On is primitive recursive, $+$ for $\langle M, \varepsilon^*, =^*\rangle$ is also primitive recursive. We denote it as $+^*$. So let $Exp(f)$ be the predicate expressing that f is an initial piece of higher exponential functions on $\langle M, \varepsilon^*, =^*\rangle$ with all needed induction information to compute any of f's values, i.e., let $Exp(f) =_{df}$

$Fcn_3(f)$ – f is a 3-ary function

$\wedge\ (\forall z)_{Ord(z)}(\forall x)_{Ord(x)}(\forall y)_{Ord(y)}((\exists w)_{Ord(w)}$

$(w = f(x, y, z)^*) \rightarrow (\forall y')_{\varepsilon^* y}(\forall z')_{\varepsilon^* z}(\exists w')_{Ord(w')}(\exists w'')_{Ord(w'')}$

$(w' = f(x, y', z)^* \wedge w'' = f(w', x, z')^*))$ – initial segment clause

$\wedge\ (\forall x)(\forall y)\forall z)(\forall w)(w = f(x, y, z)^* \rightarrow Ord(x) \wedge Ord(y)$

$\wedge\ Ord(z) \wedge Ord(w))$ – range and domains in ordinals

$\wedge\ (\forall x)_{Ord(x)}(\forall y)_{Ord(y)}(\exists w)(w = f(x, y, 0)^* \rightarrow w = x +^* y)$

for $z = 0$, $(x, y, z) = x + y$

$$\wedge\ (\forall x)_{Ord(x)}(\forall y)_{Ord(y)}(\forall z)_{Ord(z)\wedge \neq 0}(\forall w)(w = f(x, y, z)^*$$
$$\rightarrow ((\forall z')_{\varepsilon^* z}(\forall y')_{\varepsilon^* y}(f(f(x, y', z)^*, x, z')^* \varepsilon^* w \vee = w)$$
$$\wedge \neg (\exists w')_{\varepsilon^* w}(\forall z')_{\varepsilon^* z}(\forall y')_{\varepsilon^* y}(f(f(x, y', z)^*, x, z')^* \varepsilon^* w' \vee = w')))$$
$$\text{for } z \neq 0,\ f(x, y, z) = \lim_{y'<y, z'<z} f(f(x, y', z), x, z')\ .$$

Clearly this is arithmetic.

So $O(\alpha, \beta, \gamma) = \delta \leftrightarrow (\exists f)(\exists x)(\exists y)(\exists z)(\exists w)(\mathrm{Exp}(f) \wedge h(x) = \alpha \wedge h(y) = \beta \wedge h(z) = \gamma \wedge h(w) = \delta \wedge w = f(x, y, z)^*)$ is arithmetic.

Hence by Lemma 9c, O^δ is arithmetic if δ is definable.

THEOREM 14. [5]. $\langle On, <, O^\gamma \rangle \equiv \langle \mu(\omega^\omega, O_\gamma), <, O^\gamma \rangle$

Proof. [5]. ($\mu(\alpha, O_\gamma)$ is defined as

$$\mu(0, O_\gamma) = 0$$
$$\mu(\alpha + 1, O_\gamma) = (\mu\beta)_{>\mu(\alpha, O_\gamma)}(\forall \delta_1)(\forall \delta_2)(\delta_1, \delta_2 < \beta \rightarrow O_\gamma(\delta_1, \delta_2) < \beta)$$
$$\mu(\lambda, O_\gamma) = \lim_{\alpha<\gamma} \mu(\alpha, O_\gamma)\ ,$$

i.e., $\mu(\alpha, O_\gamma)$ is the α-th critical (or main) value of O_γ).

COROLLARY 15. *Say there is a standard model of*

$$ZF,\ (\exists!\, x)(F(x) \wedge Ord(x)),\ V = L\ .$$

Then there are models $\mathfrak{A}_1, \mathfrak{A}_2 \models ZF,\ (\exists!\, x)(F(x) \wedge Ord(x))$, *and* $a_i \in \mathfrak{A}_i$ *such that*:

(i) $\mathfrak{A}_i \models F(a_i) \wedge Ord(a_i)$.
(ii) $\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}, O_\zeta^{\mathfrak{A}_1} \rangle_{\zeta \leqq a_1} \approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}, O_\zeta^{\mathfrak{A}_2} \rangle_{\zeta \leqq a_2}$.
(iii) $\langle L_{\mathfrak{A}_1}, \varepsilon_{\mathfrak{A}_1} \rangle \not\approx \langle L_{\mathfrak{A}_2}, \varepsilon_{\mathfrak{A}_2} \rangle$.

Proof. Let $ZF^* = ZF \cup \{(\exists!\, x)(F(x) \wedge Ord(x))\}$. We reason as follows in $ZF^* \cup \{V = L\}$.

Let α be the unique ordinal satisfying F. So there is a unique ordinal β satisfying the predicate $x = \mu(\omega^\omega, O_\alpha)$, i.e., satisfying the predicate $(\exists \gamma)(F(\gamma) \wedge x = \mu(\omega^\omega, O_\gamma)) = G(x)$.

Furthermore, given n-Player II has a winning strategy in

$$H_n(\langle On, <, O^\alpha \rangle, \langle \mu(\omega^\omega, O_\alpha), <, O^\alpha \rangle)$$

by proof of Theorem 14.

So $\vdash_{ZF^* \cup \{V=L\}} (\exists!\, x)(G(x) \wedge Ord(x))$. So if

$$\widetilde{ZF} = ZF^* \cup \{V = L, \exists!\, x(G(x) \wedge Ord(x))\}$$

then $\widetilde{ZF}$ has a standard model.

Also $\forall n$, $\vdash_{ZF^* \cup \{V=L\}}$ player II has a winning strategy in

$$H_n(\langle On, <, O^\alpha \rangle, \langle \mu(\omega^\omega, O_\alpha), <, O^\alpha \rangle)$$

where $F(\alpha)$ and $Ord(\alpha)$.

So $\forall n$, $\vdash_{\widetilde{ZF}}$, this fact. These are the hypotheses of Lemma 12.

REMARK. Corollary 15 may be improved for O_ζ, $\zeta < \gamma$ where γ is a recursive ordinal. (The subsequent notation and results in recursion theory to be used in this remark all appear in [13].) In that event we may weaken the hypothesis of Corollary 15 to the assumption of the existence of an ω-model of ZF.

As γ is recursive, there exists $<_R$, a recursive linear order order-isomorphic to $< \upharpoonright \alpha$. In particular as $<_R$ is recursive, $<_R^{\mathfrak{A}} = <_R$ for every ω-model $\mathfrak{A}$ of analysis and hence of set theory. Let

$$F(x) = (< \upharpoonright x \approx <_R) \wedge Ord(x)) .$$

Let $\mathfrak{A}$ be an ω-model of ZF. So $\mathfrak{A} \models (\exists!\, x)F(x)$. Say $A \models F[a]$ ($a \varepsilon \mathfrak{A}$). By Theorem 14,

$$\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, O^a_{\mathfrak{A}} \rangle \equiv \langle \mu(\omega^\omega, O_{a,\mathfrak{A}}), <_{\mathfrak{A}}, O^a_{\mathfrak{A}} \rangle .$$

Now $\mu(\omega^\omega, O_{a,\mathfrak{A}})$ is itself a recursive ordinal. This is best seen by using Doner and Tarski's result that

$$\mu(\omega^\omega, O_{2\gamma}) = \mu(\omega^\omega, O_{2\gamma+1}) = \omega O_{2\gamma+2} \omega^\omega .$$

Hence to show that $\mu(\omega^\omega, O_\gamma)$ is recursive, it suffices to show that $\alpha O_\gamma \beta$ is recursive for α, β, γ recursive. This is best done by defining $O(\alpha, \beta, \gamma)$ on O, the universal system of notations.

Let $g(a, x, y, z) =$

$x +_O y$	if $z = 1$
1	if $z \neq 1, y = 1$
x	if $z \neq 1, y = 2$
$\varphi_a(\varphi_a(x, v, z), x, w)$	if $z = 2^w, y = 2^v$
$3 \cdot 5^{e_1}$	if $z = 3 \cdot 5^e, y = 2^v$

where $\varphi_{e_1} = (\lambda n)(\varphi_a(\varphi_a(x, v, z), x, \varphi_e(n)))$

$3 \cdot 5^{e_2}$	if $y = 3 \cdot 5^e$

where $\varphi_{e_2} = (\lambda n)(\varphi_a(x, \varphi_e(n), y))$

0	Otherwise.

So g is recursive. So there is a recursive function h such that

$$g(a, x, y, z) = \varphi_{h(a)}(x, y, z) .$$

By the recursion theorem there is an n such that $\varphi_n = \varphi_{h(n)}$. Let $f = \varphi_n$.

So $f(x, y, z) = x +_O y$ if $z = 1$

1 if $z \neq 1, y = 1$

x if $z \neq 1, y = 2$

$f(f(x, v, z), x, w)$ if $z = 2^w, y = 2^v$

$3 \cdot 5^{e_1}$ if $z = 3 \cdot 5^e, y = 2^v$

$$\text{where } \varphi_{e_1} = (\lambda n)(f(x, v, z), x, \varphi_e(n))$$

$3 \cdot 5^{e_2}$ if $y = 3 \cdot 5^e$

$$\text{where } \varphi_{e_2} = (\lambda n)(f(x, \varphi_e(n), y))$$

0 Otherwise.

By another theorem of Doner and Tarski we have,

$\alpha O_0 \beta = \alpha + \beta$

$\alpha O_\gamma 0 = 0$ if $\gamma \geqq 1$

$\alpha O_\gamma 1 = \alpha$ if $\gamma \geqq 1$

$\alpha O_{\gamma+1}(\beta + 1) = (\alpha O_{\gamma+1}\beta) O_\gamma \alpha$

$\alpha O_\lambda(\beta + 1) = \lim_{\gamma<\lambda}(\alpha O_\lambda \beta) O_\gamma \alpha$ if $\lim(\lambda)$

$\alpha O_\gamma \lambda = \lim_{\beta<\lambda}(\alpha O_\gamma \beta)$ if $\lim(\lambda)$.

So by induction on O, we have if $x, y, z \in O$, then

$$|f(x, y, z)|_O = |x|_O O_{|z|_O} |y|_O \text{ and } f(x, y, z) \in O .$$

So if α, β, γ are recursive ordinals, then $\alpha O_\gamma \beta$ is a recursive ordinal. In particular $\mu(\omega^\omega, O_{a,\mathfrak{A}})$ is a recursive ordinal. Let $<_S$ be a recursive linear ordering such that $<_{\mathfrak{A}} \restriction O_{a,\mathfrak{A}} \approx <_S$. Then define $O^a_{\mathfrak{A}}$ according to $<_S$ by induction on $<_S$. It has an arithmetic definition. So as

$$\langle \mu(\omega^\omega, O_{a,\mathfrak{A}}), <_{\mathfrak{A}}, O^a_{\mathfrak{A}} \rangle \approx \langle \mathrm{Dom}\,(<_S), <_S, O^a_{\mathfrak{A}} \text{ according to } <_S \rangle ,$$

$\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, O^a_{\mathfrak{A}} \rangle \equiv \langle \mathrm{Dom}\,(<_S), <_S, O^a_{\mathfrak{A}}$ according to $<_S\rangle$ which has a Δ^1_1 truth-set as $\mathrm{Dom}\,(<_S), <_S, O^a_{\mathfrak{A}}$ according to $<_S$ are all arithmetic.

On the other hand, as $\mathfrak{A}$ is an ω-model of set theory, $P(\omega)_{\mathfrak{A}}$ is an ω-model of analysis and $Th_{P(\omega)_{\mathfrak{A}}} \leqq_T Th_{\mathfrak{A}}$. But the former is not Δ^1_1 and so neither is $Th_{\mathfrak{A}}$ or $Tr_{\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, \mathrm{Arith}_{\mathfrak{A}}\rangle}(<, O^a, \tilde{\varepsilon})$. So as in the proof of Corollary 8, if there is an ω-model $\mathfrak{A}$ of set theory, then there are models $\mathfrak{A}_1, \mathfrak{A}_2 \models ZF, \exists!\, xF(x)$ and ordinals $a_i \in \mathfrak{A}_i$ such that

(i) $\mathfrak{A}_i \models F[a_i]$

(ii) $\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}, O^{\mathfrak{A}_1}_\zeta \rangle_{\zeta \leqq a_1} \approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}, O^{\mathfrak{A}_2}_\zeta \rangle_{\zeta \leqq a_2}$

(iii) $\langle L_{\mathfrak{A}_1}, \varepsilon_{\mathfrak{A}_1} \rangle \not\approx \langle L_{\mathfrak{A}_2}, \varepsilon_{\mathfrak{A}_2} \rangle$

(provided that in any ω-model, the unique ordinal α such that $F(\alpha)$ is recursive).

Let γ be a definable ordinal. Given an arithmetic function f we define f_γ as follows:

$$f_\gamma(\alpha_1 \cdots \alpha_n) = \begin{cases} f(\alpha_1 \cdots \alpha_n) \text{ if } \alpha_1 \cdots \alpha_n, f(\alpha_1 \cdots \alpha_n) < \gamma \\ 0 \text{ otherwise.} \end{cases}$$

So by Lemma 9c, each f_γ is arithmetic.

Let $Arith_\gamma = \{f_\gamma | f \in Arith\}$.

We now proceed to show that appropriate ordinals can be found so that we can show $\langle \alpha, <, A', Arith_\gamma \rangle \equiv \langle On, <, A', Arith_\gamma \rangle$ for various $A' \subseteqq Arith$. Once if we have shown this (provided that α is provably less than On, so that it is less than On in all models) we immediately can obtain not determined up to $\equiv$ results as in Corollary 15 above.

In order to obtain such appropriate α we must briefly consider H_n in greater detail. H_n is an n move finite 2 person game. At move i, player 1 choses a model $\mathfrak{A}_{l(i)}$, an integer $k(i)$ and $k(i)$ points $a_{1i}^{l(i)} \cdots a_{k(i)i}^{l(i)}$ in model $\mathfrak{A}_{l(i)}$. Player 2 responds by choosing $k(i)$ points $a_{k(i)i}^{3-l(i)}$ in the other model. At the end Player 2 is said to win just in case the correspondence $a_{ji}^1 \leftrightarrow a_{ji}^2$ $i = 1 \cdots n$, $j = 1 \cdots k(i)$ is a partial isomorphism (with respect to the relations and operations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$).

So in a game $H_n(\langle \alpha, <, A' \rangle, \langle On, <, A' \rangle)$ if 2's winning strategy can be constructed to preserve ordinals $<\gamma$, then it is clear that the partial isomorphism will also extend to $Arith_\gamma$.

LEMMA 16. *Player 2 has a winning strategy in* $H_n(\langle \mu(\gamma + \omega^\omega), < \rangle)$ *preserving ordinals* $<\gamma$.

Proof. 2's strategy is as follows:

For ordinals $<\gamma$ he leaves them fixed. For the ω^ω segment between γ and $\gamma + \omega^\omega$ "versus" the On segment $\geqq \gamma$, 2 uses his winning strategy from $H_n(\langle \omega^\omega, < \rangle, \langle On, < \rangle)$ "shifted over by γ".

Let $\gamma^* = \lim_{\substack{n<\omega \\ \alpha<\gamma}} \omega^\alpha \cdot n$. Then clearly $\gamma^* \geqq \gamma$ (for $\gamma > 0$).

LEMMA 17. *Player 2 has a winning strategy in*

$$H_n(\langle \omega^{\gamma+\omega^\omega}, <, + \rangle, \langle On, <, + \rangle)$$

preserving ordinals $<\gamma^*$.

Proof. One simply observes that in the winning strategy for 2 defined by Ehrenfeucht in this game from the winning strategy for 2 in the game $H_n(\langle \gamma + \omega^\omega, < \rangle, \langle On, < \rangle)$ if ordinals $<\gamma$ are preserved in the latter game, then ordinals $<\gamma^*$ are preserved in the former.

LEMMA 18. *Player 2 has a winning strategy in*

$$H_n(\langle \omega^{\omega^{\gamma+\omega^\omega}}, <, +, \times \rangle, \langle On, <, +, \times \rangle)$$

preserving ordinals $<\gamma^{**}$.

Proof. As in Lemma 17.

THEOREM 19.

$$\langle \omega^{\omega^{\gamma+\omega^\omega}}, <, +, \times, Arith_\gamma \rangle \equiv \langle On, < +, \times, Arith_\gamma \rangle .$$

Proof. As described above.

COROLLARY 20. *Say there is a standard model of*

$$ZF, (\exists!\, x)(F(x) \wedge Ord(x)),\ V = L.$$

Then there are models $\mathfrak{A}_1, \mathfrak{A}_2 \models ZF$, $a_i \in A_i$ *such that*
(i) $\mathfrak{A}_i \models F(a_i) \wedge Ord(a_i)$
(ii) $\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}, +_{\mathfrak{A}_1}, \times_{\mathfrak{A}_1}, \mathrm{Arith}_{a_1}^{\mathfrak{A}_1} \rangle$
$\approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}, +_{\mathfrak{A}_2}, \times_{\mathfrak{A}_2}, Arith_{a_2}^{\mathfrak{A}_2} \rangle$
(iii) $L_{\mathfrak{A}_1} \not\equiv L_{\mathfrak{A}_2}$.

Proof. As in Corollary 15.

REMARKS. (1) If $F(x)$ is a predicate of the form $< \restriction x \approx <_R$ for some recursive well-ordering $<_R$, then the hypothesis of corollary 20 may be weakened as were those of corollary 15 to supposing the existence of ω-models of ZF.

(2) Doner claims that Lemma 18 may be extended to the statement that 2 has a winning strategy in

$$H_n(\langle \mu(\delta + \omega^\omega, O^\gamma), <, O^\gamma \rangle, \langle On, <, O^\gamma \rangle)$$

preserving ordinals $<\delta$. Hence we may conclude by similar arguments to those above that the constructible sets are not determined up to $\equiv$ by $<$, $Arith_\delta$, O^γ where δ, γ are definable ordinals. Again, as usual the hypothesis may be weakened if both δ, γ are recursive ordinals.

(3) The pairing function p of Godel [8] can be shown by a long tedious computation to be definable in terms of $+, \times$. Hence in any result where we have shown that the structure of L is not de-

termined up to $\equiv$ by $<, A'$ where $+, \times \varepsilon A'$ we may conclude that the structure of L is not determined up to $\equiv$ by $<, A' \cup \{p\}$. Feferman has pointed out to us that the pairing function $q(\alpha, \beta) = 2^{\alpha}(\beta + 1)$ is immediately definable in terms of $+$, $\times$ and exponentiation (O_2), and hence if the structure of L is not determined up to $\equiv$ by $<, A'$ where $O_0, O_1, O_2 \varepsilon A'$ then A' might as well contain q.

(4) As $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, +_{\mathfrak{A}}, \times_{\mathfrak{A}}\rangle \equiv \omega_{\mathfrak{A}}^{\omega^{\omega^{\omega}}}, <_{\mathfrak{A}}, +_{\mathfrak{A}}, \times_{\mathfrak{A}}\rangle$ and as

$$Th_{\langle \omega_{\mathfrak{A}}^{\omega^{\omega^{\omega}}}, <_{\mathfrak{A}}, +_{\mathfrak{A}}, \times_{\mathfrak{A}}\rangle}$$

is not recursive for any model $\mathfrak{A}$ of ZF, we may conclude that

$$Tr_{\langle On, <\cdot Arith\rangle}(<, +, \times) \not\leq_T Tr_{\langle On, <, Arith\rangle}(<) .$$

Hence if ZF is consistent, then there are models $\mathfrak{A}_1, \mathfrak{A}_2$ of ZF with $\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}\rangle \approx \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}\rangle$ but

$$\langle On_{\mathfrak{A}_1}, <_{\mathfrak{A}_1}, +_{\mathfrak{A}_1}, \times_{\mathfrak{A}_1}\rangle \not\equiv \langle On_{\mathfrak{A}_2}, <_{\mathfrak{A}_2}, +_{\mathfrak{A}_2}, \times_{\mathfrak{A}_2}\rangle .$$

Identically we may conclude that if ZF is consistent then $+^{\omega}$, x^{ω} are not determined up to $\equiv$ by $<$ where

$$x^{\omega}(\alpha, \beta) = \begin{cases} \alpha + \beta \text{ if } \alpha, \beta < \omega \\ 0 \text{ otherwise} \end{cases}$$

$$x^{\omega}(\alpha, \beta) = \begin{cases} \alpha \cdot \beta \text{ if } \alpha, \beta < \omega \\ 0 \text{ otherwise} \end{cases}$$

and hence likewise the structure of L is not determined up to $\equiv$ by $<, +^{\omega}, x^{\omega}$ (if ZF has ω-models). Now let HFS be the hereditarily finite sets. Then it is easily seen that there is a one-to-one onto definable map of HFS into ω such that the image of ε and $=$ are arithmetic in the sense of recursion theory and hence in particular definable in terms of $+_{\omega}, \times_{\omega}$. Also $+_{\omega}, \times_{\omega}$ are definable in terms of ε on HFS simply by the use of their recursive definitions. Hence if ZF is consistent then the structure of HFS is not determined up to $\equiv$ by $<$, and if ZF has ω-models then the structure of L is not determined up to $\equiv$ by the structure of HFS.

(5) On the other hand, as $\tilde{\varepsilon}$ gives us in $\langle On_{\mathfrak{A}}, <, Arith_{\mathfrak{A}}\rangle$ (for any model $\mathfrak{A}$ of ZF) a model of ZF we may in this model give explicit definitions of all primitive recursive functions which are of course definable in terms of $\tilde{\varepsilon}$ and hence arithmetic and as h the function mapping ordinals of this model isomorphically to $On_{\mathfrak{A}}$ is also primitive recursive, we can pull these functions over to arithmetic functions on $On_{\mathfrak{A}}$. By the axiom schema of transfinite induction in ONT' these are the original primitive recursive functions. So if $i: \langle On_1, <_1, \tilde{\varepsilon}_1\rangle \underset{\rightarrow}{\approx} \langle On_2, <_2, \tilde{\varepsilon}_2\rangle$ then let $i' = h_2 i h_i^{-1}$ (where $h_i: On_{\langle On_i\ \tilde{\varepsilon}_i \cdot \cong_i\rangle} \underset{\rightarrow}{\approx} On_i$) and one

may immediately conclude that i': $\langle On_1, Arith_1\rangle \overset{\approx}{\rightarrow} \langle On_2, Arith_2\rangle$. So all arithmetic functions are determined up to $\approx$ by $\tilde{\varepsilon}$.

(6) Finally we observe, if we restrict the question if the structure of L is determined up to $\approx$ by $<$ merely to standard models, then the answer, as is well-known, is yes because to show the L's are isomorphic we have available "real" transfinite induction in the "real" world, i.e., in our metalanguage outside our models.

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