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ON A PROBLEM OF DANZER

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By a Danzer set S we shall mean a subset of the n-dimensional Euclidean space R_n which has the property that every closed convex body of volume one in R_n contains a point of S. L. Danzer has asked if for $n \geqq 2$ there exist such sets S with a finite density. The answer to this question is still unknown. In this note our object is to prove two theorems about Danzer sets.

If Λ is a n-dimensional lattice, any translate $\Gamma = \Lambda + p$ of Λ will be called a grid Γ; Λ will be called the lattice of Γ and the determinant $d(\Lambda)$ of Λ will be called the determinant of Γ and will be denoted by $d(\Gamma)$. In § 2 we prove

THEOREM 1. For $n \geqq 2$, a Danzer set cannot be the union of a finite number of grids.

Let S be a Danzer set and $X > 0$ a positive real number. Let $N(S, X)$ be the number of points of S in the box $\max_{1 \leqq i \leqq n} |x_i| \leqq X$. Let $D(S, X) = N(S, X)/(2X)^n$. In § 3 we prove

THEOREM 2. There exist Danzer sets S with $D(S, X) = 0((\log X)^{n-1})$ as $X \rightarrow \infty$.

The case $n = 2$ of the theorem is known, although no proof seems to have been published. The referee has pointed out that a lower bound of 2 can easily be established for the density of a Danzer set in $n = 2$, but the authors are unaware of any further results in this direction.

2. Proof of Theorem 1. We shall assume throughout that $n \geqq 2$. It is obvious that if S is a Danzer set and T is a volume preserving affine transformation of R_n onto itself, then $T(S)$ is also a Danzer set.

Let $S_1, S_2, \cdots$ be a sequence of sets in R_n. Let S be the set of points X such that there exists a subsequence $S_{i_1}, S_{i_2}, \cdots$ of $\{S_r\}$ and points $X_{i_r} \in S_{i_r}$, such that $X_{i_r} \rightarrow X$ as $r \rightarrow \infty$. We write

$$S = \lim_{r \rightarrow \infty} S_r = \lim S_r .$$

LEMMA 1. *Let $\{S_r\}$ be a sequence of Danzer sets in R_n. Then $S = \lim S_r$ is also a Danzer set.*

Proof. Let K be a closed convex body of Volume 1. Then for each r, $K \cap S_r \neq \phi$, so that for each r, there exists $X_r \in K \cap S_r$. Since K is compact, $\{X_r\}$ has a convergent subsequence $\{X_{i_r}\}$ converging to a point X in $K \cap S$.

LEMMA 2. *Let* $S^{(j)} = \lim_{r\to\infty} S_r^{(j)}$, $j = 1, \cdots, k$. *Then*

$$\bigcup_{j=1}^{k} S^{(j)} = \lim_{r\to\infty}\left(\bigcup_{j=1}^{k} S_r^{(j)}\right).$$

Proof. $X \in \cup S^{(j)} \Rightarrow X \in S^{(j)}$ for some j, say $j = j_0 \Rightarrow$ there exist a subsequence $\{S_{i_r}^{(j_0)}\}$ of $\{S_{i_r}^{(j_0)}\}$ and points $X_{i_r} \in S_{i_r}^{(j_0)}$ such that $X_{i_r} \to X \Rightarrow X_{i_r} \in \cup S_{i_r}$ and $X_{i_r} \to X \Rightarrow X \in \lim_{r\to\infty} (\bigcup_{j=1}^{k} S_r^{(j)})$. Thus $\cup S^{(j)} \subset \lim (\bigcup_{j=1}^{k} S_r^{(j)})$. Let $X \in \lim (U_{j=1}^{k} S_r^{(i)})$. Then there exists a sequence $\{i_r\}$ of natural numbers and $X_{i_r} \in \bigcup_{j=1}^{k} S_{i_r}^{(j)}$ such that $X_{i_r} \to X$. Since k is finite, there exists a $j = j_0$ say, and an infinite subsequence k_r of i_r such that $X_{k_r} \in S_{k_r}^{(j_0)}$. Then $X_{k_r} \to X$ and $X \in S^{(j_0)}$, so that $X \in \cup S^{(j_0)}$ and $\lim_{r\to\infty} (\bigcup_{j=1}^{k} S_s^{(j)}) \subset \cup S^{(j)}$.

This completes the proof of the lemma.

LEMMA 3. *Let $\Gamma_1, \Gamma_2, \cdots$ be a sequence of grids in R_n with equal determinants $d(\Gamma_r) = \Delta$. Then $\{\Gamma_r\}$ has a subsequence $\{\Gamma_{i_r}\}$, such that $\lim_{r\to\infty} \Gamma_{i_r}$ is either a grid or is contained in a hyperplane.*

Proof. If $\lim_{r\to\infty} \Gamma_r = \phi$, there is nothing to prove. Assume, therefore, that $\Gamma = \lim_{r\to\infty} \Gamma_r \neq \phi$. Let $X \in \Gamma$. Then there exists a subsequence $\{i_r\}$ of natural numbers and points $X_{i_r} \in \Gamma_{i_r}$, such that $X_{i_r} \to X$. Then $\Lambda_{i_r} = \Gamma_{i_r} - X_{i_r}$ is a sequence of homogeneous lattices and $\lim \Gamma_{i_r} = X + \lim \Lambda_{i_r}$. Therefore, it is enough to prove the theorem for lattices.

Let $\{\Lambda_r\}$ be a sequence of lattices with determinants $d(\Lambda_r) = \Delta$, independent of r. Let $\mu_1(\Lambda_r), \cdots, \mu_n(\Lambda_r)$ be the successive minima of the Euclidean distance with respect to Λ_r, i.e., $\mu_i(\Lambda_r) = \inf \mu$: such that $|X| < \mu$ has i linearly independent points of Λ_r.

Suppose, first, that there exists $\delta > 0$, such that $\mu_1(\Lambda_r) \geqq \delta$ for infinitely many r. Then a subsequence satisfies the conditions of Mahler's compactness theorem and has a subsequence convergent in the sense of Mahler (see, e.g., Cassels [2]). The last subsequence converges to the limiting lattice in our sense also.

We may, therefore, assume $\mu_1(\Lambda_r) \to 0$ as $r \to \infty$. Since

$$\mu_1(\Lambda_r) \cdots \mu_n(\Lambda_r) \geqq \frac{2^n}{n!} \cdot \frac{1}{J_n},$$

where J_n is the volume of the sphere $|X| < 1$, (see, e.g., Cassels [2]), and since $n \geqq 2$, it follows that $\mu_n(\Lambda_r) \to \infty$ as $r \to \infty$. For each r, let $P_{r_1}, \cdots, P_{r_n}$ be points such that $|P_{r_i}| = \mu_i(\Lambda_r)$. Let π_r be the plane through $0, p_{r_1}, \cdots, p_{r_{n-1}}$. It is easily seen that there exists a subsequence $\{\Lambda_{i_r}\}$ of $\{\Lambda_r\}$ such that the sequence $\{\pi_{i_r}\}$ converges to a plane π. We assert that $\lim_{r\to\infty}\{\Lambda_{i_r}\} \subset \pi$. For, let $X \in \lim_{r\to\infty} \Lambda_{i_r}$. Then $X = \lim X_{k_r}$, where k_r is a subsequence of i_r and $X_{k_r} \in \Lambda_{k_r}$. There exists M independent of k_r, such that $|X_{k_r}| \leqq M$ for all k_r. Also

$$X_{k_r} = g_{r,1}P_{k_r,1} + \cdots + g_{r,n}P_{k_r,n},\ g_{r,i}\ \text{real}\ ,$$

and if $g_{r,n} \neq 0$ then $|X_{k_r}| \geqq \mu_n(\Lambda_{k_r})$. Since $\mu_n(\Lambda_{k_r}) \to \infty$ as $r \to \infty$, $g_{r,n} = 0$ for all large r and $X \in \pi$. This proves the lemma

LEMMA 4. *Let $\{\pi_i\}$ be a sequence of hyperplanes. Then $\{\pi_i\}$ has a subsequence $\{\pi_{i_\mu}\}$ whose limit lies in a hyperplane.*

Proof. If $\pi = \lim_{i\to\infty} \pi_i = \phi$ then there is nothing to prove. Assume, therefore, $X \in \pi$. Then there is a subsequence $\{k_r\}$ of natural numbers and points $X_{k_r} \in \pi_{k_r}$ such that $X_{k_r} \to X$. The planes $\hat{\pi}_{k_r} = \pi_{k_r} - X_{k_r}$ pass through 0 and have a subsequence $\hat{\pi}_{i_r}$ which converges to a plane $\hat{\pi}$ say. Then $\lim_{r\to\infty} \pi_{i_r} = \hat{\pi} + X$. This proves the lemma.

Proof of Theorem 1. We shall prove more, namely, *a Danzer set cannot be the union of a finite number of hyperplanes and a finite number of grids.*

Let $S = \bigcup_{i=1}^{r} \pi_i \bigcup_{j=1}^{t} \Gamma_j$ be a Danzer set, such that π_i are hyperplanes and Γ_j are grids. Let $t \geqq 1$. Let $X \neq Y$, $X, Y \in \Gamma_1$. For each positive integer k, let T_k be a volume preserving affine transformation such that $T_k(X) = X$ and $|T_k(Y) - X| = k^{-1}|Y - X|$. Since $n \geqq 2$, such transformations exist. For each k, $T_k(S)$ is a Danzer set, and by Lemma 1, so is the limit of every subsequence of $\{T_k(S)\}$. By Lemmas 3 and 4 we can choose a subsequence $\{T_{k_r}\}$ of $\{T_k\}$ such that each $\lim_{t\to\infty} T_{k_r}(\pi_i)$ lies in a hyperplane, while each $\lim_{t\to\infty} T_{k_r}(\Gamma_j)$ is either a grid or lies in a hyperplane. Since

$$\lim_{r\to\infty} T_{k_r}(S) = \bigcup_{i=1}^{t} \lim T_{k_r}(\pi_i) \bigcup_{j=1}^{t} \lim T_{k_r}(\Gamma_j)$$

and $\lim T_{k_r}(\Gamma_1)$ is in a hyperplane, the Danzer set $\lim T_{k_r}(S)$ lies in the union of a finite number of hyperplanes and $t_1 < t$ grids, so that we have (by increasing $T_{k_r}(S)$ if necessary) a Danzer set consisting of a finite number of hyperplanes and $t_1 < t$ grids. Repeating this process a number of times we obtain a Danzer set that is the union of a finite number of hyperplanes. This can easily be seen to lead to a contradiction which proves the theorem.

3. Proof of Theorem 2. Let K be a closed convex body in R_n. The set $S \subset R_n$ is said to be a covering set for K if $R_n \subset \bigcup_{A \in S}(K + A)$. The set S contains a point of each translate of K if and only if S is a covering set for $-K$. Clearly a set S is a Danzer set if and only if it is a covering set for each closed convex body of volume one. Therefore, in order to prove a given set S is a Danzer set, it is enough to prove that for every closed convex body K of volume one, S contains a covering set for K.

If Γ is a grid with lattice Λ, then it is easy to see that Γ is a covering set for K if and only if Λ is.

Let π be a parallelepiped. Let A_0 be one of its vertices and $A_1, \cdots, A_n$ be the n vertices joined to A_0 by edges of π. Let Λ be the lattice generated by $A_1 - A_0, \cdots, A_n - A_0$. By the grid generated by π we shall mean the grid $\Lambda + A_0$. It is easily seen that if a closed convex body K contains a parallelepiped which generates a grid Γ, then Γ is a covering set for K.

A lattice Λ will be called rectangular if it consists of points $(\alpha_1 u_1, \cdots, \alpha_n u_n)$, where α_i are fixed positive real numbers and u_i take integral values. A grid Γ will be called rectangular if its lattice is rectangular.

Let $\alpha_1, \cdots, \alpha_n$ be positive real numbers. Let Γ_α be the grid generated by the parallelepiped $|x_i| \leqq \alpha_i$. Let B be a box $|x_i| \leqq \beta_i$, where $\beta_i \geqq \alpha_i$ for $i = 1, \cdots, n$. Then Γ_α is clearly a covering set for B.

Let K be a closed convex body of volume one. Let K_1 be the steiner symmetrical of K with respect to the plane $x_1 = 0$. Let K_2 be the steiner symmetrical of K_1 with respect to $x_2 = 0$ and so on. Then K_n is symmetrical about all the coordinate planes and has volume one. We next have

LEMMA 5. *If a rectangular lattice Λ is a covering set for K_n, then it is a covering set for K also.*

(The lemma and its proof are easy adaptions of Lemma 2 of Sawyer (3). For completeness, we give the proof below).

Proof. Let Λ be the rectangular lattice consisting of points $(\alpha_1 u_1, \cdots, \alpha_n u_n)$, $\alpha_i > 0$ fixed real numbers and u_i running over the set of integers. It is enough to prove that if Λ is a covering set for K_1, then it is a covering set for K also.

Let $\Lambda_1 =$ subset of Λ in the plane $x_1 = 0$. The sets $K_1 + \Lambda$ cover R_n. We assert each line $x_2 = a_2, \cdots, x_n = a_n$ meets $K_1 + P$ is a segment of length at least α_1 for some $P \in \Lambda_1$. Such a line meets only a finite number of translates $K_1 + P_s$, $P_s \in \Lambda_1$, each of them in a segment $|x_1| \leqq b_s$ and hence meets $K_1 + \Lambda_1$ in the segment $|x_1| \leqq b = \max b_s$. If $b < \frac{1}{2}\alpha_1$, then $K_1 + \Lambda$ meets the line in segments $|x_1 - m\alpha_1| \leqq b < \frac{1}{2}\alpha_1$, where m takes integral values. This leaves part of the line uncovered by sets $K_1 + \Lambda$, contrary to the fact that Λ is a covering set for K_1. Thus $b \geqq \frac{1}{2}\alpha_1$, i.e., $b_s \geqq \frac{1}{2}\alpha_1$ for some s. Therefore, the line meets $K_1 + P_s$ and hence $K + P_s$ in a segment of length at least α_1, and is therefore, covered by the sets $K + \Lambda$. Since this is true for all such lines, Λ is a covering set for K.

COROLLARY. *A rectangular grid Γ which is a covering set for K_n is also a covering set for K.*

Because of the corollary, in oder to prove that a given set S is a Danzer set, it is enough to prove that for every given closed convex body K of volume one, which is symmetrical about all the coordinate planes, S contains a rectangular grid Γ which is a covering set for K.

Let K be a closed convex body of volume one, which is symmetrical about the coordinate planes. Then K contains a point $(a_1, \cdots, a_n)$, $a_i > 0$, such that $2^n a_1 \cdots a_n \geqq n!/n^n$. (See, e.g., Sawyer [3]). Then K contains a box $B_\beta\colon |x_i| \leqq \beta_i$, $\beta_i \leqq a_i$ with volume $2^n\beta_1 \cdots \beta_n = n!/n^n$. A covering rectangular grid of B_β is automatically a covering set for K. Therefore, S is a Danzer set if for all closed boxes B_β of volume $n!/n^n$, S contains a rectangular grid Γ_α generated by $|x_i| \leqq \alpha_i$ with $\alpha_i \leqq \beta_i$.

We now construct a set A of points $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i > 0$, such that for each set $(\beta_1, \cdots, \beta_n)$, $\beta_i > 0$, $\beta_1 \cdots \beta_n = n!/(2n)^n = k$, say, there exists an $\alpha \in A$, such that $\alpha_i \leqq \beta_i$. Then the grid Γ_α will provide a convering by B_β and the set $S = \bigcup_{\alpha \in A} \Gamma_\alpha$ will be a Danzer set.

Let H be the set of point x such that $x_1 \cdots x_n = k$, $x_i > 0$. Divide the part $x_1 > 0, \cdots, x_{n-1} > 0$ of the plane $x_n = 0$ into $n-1$ dimensional parallelepipeds $\pi_{k_1,\cdots,k_{n-1}}$ defined by

$$e^{k_i} \leqq x_i \leqq e^{k_i+1},\ i = 1, \cdots, n-1,\ (k_1, \cdots, k_{n-1}) \in Z^{n-1}\ ,$$

when Z is the set of rational integers. Let $H_{k_1,\cdots,k_{n-1}} = \{x\colon x \in H$ and $(x_1, \cdots, x_{n-1}) \in \pi_{k_1,\cdots,k_{n-1}}\}$. Then $H = \bigcup_{(k_1,\cdots,k_{n-1}) \in S^{n-1}} H_{k_1,\cdots,k_{n-1}}$. If $X \in H_{k_1,\cdots,k_{n-1}}$, then $x_i \geqq e^{k_i}$, $i = 1, \cdots, n-1$ and

$$x_n = \frac{k}{x_1 \cdots x_{n-1}} \geqq \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}}\ .$$

Let

$$\alpha = \alpha_{k_1,\cdots,k_{n-1}} = \left(e^{k_1}, \cdots, e^{k_{n-1}}, \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}}\right)$$

Then Γ_α is a grid of determinant $2^n k/e^{n-1}$. Let

$$A = \{\alpha_{k_1,\cdots,k_{n-1}}: (k_1, \cdots, k_{n-1}) \in Z^{n-1}\} .$$

For each $\beta = (\beta_1, \cdots, \beta_n) \in H_{k_1,\cdots,k_{n-1}}$, $\alpha_{k_1,\cdots,k_{n-1}} \in A$ has the property that Γ_α is a covering set for B_β. Therefore $S = \bigcup_{\alpha \in A} \Gamma_\alpha$ is a Danzer set. To prove Theorem 2, it will be enough to prove $D(S, X) = O((\log X)^{n-1})$, as $X \to \infty$.

Let $B(X)$ be the box $|x_i| \leqq X$. Since $N(S, X)$, $N(\Gamma_\alpha, X)$ denote the number of points of S and Γ_α, respectively, in $B(X)$, it follows that

$$N(S, X) \leqq \sum_{\alpha \in A} N(\Gamma_\alpha, X) . \tag{$*$}$$

If $\alpha = \alpha_{k_1,\cdots,k_{n-1}} \in A$, then the points of Γ_α have coordinates

$$\begin{aligned} &\left(e^{k_1}u_1, e^{k_2}u_2, \cdots, e^{k_{n-1}}u_{n-1}, \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}}u_n\right) \\ &= (e^{k_1}u_1, e^{k_2}u_2, \cdots, e^{k_{n-1}}u_{n-1}, ke^l u_n) , \end{aligned}$$

say, where u_i are odd integers. If $\Gamma_\alpha \cap B(X) \neq \phi$, then

$$e^{k_1} \leqq X, \cdots, e^{k_{n-1}} \leqq X, ke^l \leqq X ,$$

so that for

$$\begin{aligned} i = 1, 2, \cdots, n-1, e^{k_i} &\geqq \frac{k}{e^{n-1}} \cdot \frac{e^{k_i}}{e^{k_1+\cdots+k_{n-1}}} \cdot \frac{1}{X} \\ &\geqq \frac{k}{e^{n-1}} \cdot \frac{1}{X^{n-1}} . \end{aligned}$$

Therefore,

$$\begin{aligned} \Gamma_\alpha \cap B(X) \neq \phi &\Rightarrow \frac{k}{(eX)^{n-1}} \leqq e^{k_i} \leqq X, \text{ for } i = 1, \cdots, n-1 \\ &\Rightarrow \log k - (n-1)(1 + \log X) \leqq k_i \leqq \log X \\ &\qquad\qquad \text{for } i = 1, \cdots, n-1 . \end{aligned}$$

Therefore, the number $\nu(X)$ of α for which $\Gamma_\alpha \cap B(X) \neq \phi$, satisfies

$$\begin{aligned} \nu(X) &\leqq (n(1 + \log X) - \log k)^{n-1} \\ &= O(\log X)^{n-1} . \end{aligned} \tag{$**$}$$

If $\Gamma_\alpha \cap B(X) \neq \phi$, then the number $N(\Gamma_\alpha, X)$ of points of Γ_α in $B(X)$ is the number of points $(u_1, \cdots, u_n) \in Z^n$, u_i odd, with

$$-X \leqq u_i e^{k_i} \leqq X, i = 1, \cdots, n-1$$

and

$$-X \leqq u_n \frac{k}{e^{k_1+\cdots+k_{n-1}+n-1}} \leqq X .$$

Writing $[\xi]$ for the largest integer $\leqq \xi$, we have

$$\begin{aligned} N(\Gamma_\alpha, X) &= \left(\prod_{i=1}^{n-1} 2\left[\frac{1}{2}\left(\frac{X}{e^{k_i}} + 1\right)\right]\right) 2\left[\frac{1}{2}\left(\frac{Xe^{k_1+\cdots+k_{n-1}+n-1}}{k} + 1\right)\right] \\ (***) \qquad &\leqq 2^n\left(\prod_{i=1}^{n-1} \frac{X}{e^{k_i}}\right)\frac{Xe^{k_1+\cdots+k_{n-1}+n-1}}{k} \\ &= (2X)^n e^{n-1}/k \,. \end{aligned}$$

Combining $(*)$, $(**)$ and $(***)$, we get

$$D(S, X) = N(S, X)/(2X)^n = O((\log X)^{n-1}) \,.$$

Thus S is a Danzer set which provides an example for Theorem 2.

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