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ON THE DEGREE OF THE MINIMAL POLYNOMIAL OF A COMMUTATOR OPERATOR

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Let A be an n-square matrix over a field F of characteristic 0. The additive commutator operator defined by A, $T_A X = AX - XA$, can be regarded as a linear transformation on the space of all n-square matrices X over F. Following earlier papers by O. Taussky and H. Wielandt and one of the present authors, we show that the degree of the minimal polynomial of T_A is always odd and at least

$$2[m + E + (k-2)e - k] + 1$$

where m is the degree of the minimal polynomial of A, k is the number of distinct eigenvalues of A, and $E(e)$ is the largest (least) integer among the degrees of the distinct highest degree elementary divisors of the characteristic matrix of A.

The purpose of this paper is two-fold: first we obtain a count of the number of distinct differences of the form $z_i - z_j$, $i \neq j$, where $z_1, \cdots, z_n$ are distinct elements of a field F of characteristic 0; second we apply this to prove a result on the parity and magnitude of the degree of the minimal polynomial of a matrix commutator. Annihilating polynomials for commutators were originally considered by Taussky and Wielandt in a paper in 1962 [5] and then again by one of the present authors in 1964 [2] (see also [1] and [6]).

To be precise let A be an n-square matrix over F and consider the linear transformation T_A defined on the space $M_n(F)$ of n-square matrices over F:

$$T_A X = AX - XA\,, \qquad X \in M_n(F)\,. \tag{1}$$

Then T_A is called the commutator operator defined by A. The transformation T_A has a matrix representation $A \otimes I_n - I_n \otimes A$ where $\otimes$ indicates Kronecker product [3, p. 8]. The minimal polynomial of T_A is called the minimal polynomial of the commutator operator (1).

In an appropriate algebraic extension field K of F the elementary divisors of the characteristic matrix of A are powers of binomials. Let $\gamma_1, \cdots, \gamma_k$ be the distinct eigenvalues of A, let e_i be the degree of the highest degree elementary divisor of the characteristic matrix of A involving γ_i, $i = 1, \cdots, k$, let $E = \max_i e_i$, $e = \min_i e_i$, and let m be the degree of the minimal polynomial of A.

THEOREM 1. *If F is a field of characteristic zero then the degree* *of the minimal polynomial of the commutator operator* T_A *is always odd and at least*

$$2[m + E + (k-2)e - k] + 1 .$$

In order to prove Theorem 1 we shall find it necessary to consider the following problem: given n distinct numbers $z_1, \cdots, z_n$ in F how many distinct differences are there of the form $z_i - z_j$, $i \neq j$, $i, j = 1, \cdots, n$? Of course, the number can be as small as $2n-2$ by simply taking $z_i = i$, $i = 1, \cdots, n$. As an application of the Perron-Frobenius theorem on nonnegative matrices the following result, used to prove Theorem 1, may be of some independent interest.

THEOREM 2. *Let* $z_1, \cdots, z_n$ *be* n *distinct element in a field* F *of characteristic* 0. *Then there are always at least* $2n-2$ *distinct non-zero differences of the form* $z_i - z_j$, $i \neq j$, $i, j = 1, \cdots, n$.

II. *Proofs*. We begin with the proof of Theorem 2. We shall show in fact that there exists a permutation $\varphi \in S_n$ (the symmetric group of degree n) for which the $2n-2$ elements

$$\pm(z_{\varphi(1)} - z_{\varphi(2)}), \cdots, \pm(z_{\varphi(1)} - z_{\varphi(n)}) \tag{2}$$

are distinct. If this is not the case then for every $\varphi \in S_n$ there must exist integers p and q, $p \neq q$, such that

$$z_{\varphi(1)} - z_{\varphi(p)} = z_{\varphi(q)} - z_{\varphi(1)} . \tag{3}$$

For, obviously the two sets of numbers (2) obtained by choosing first the $+$signs and then the $-$signs each consist of $n-1$ distinct differences. Thus if there is to be an overlap, (3) must hold and we have $z_{\varphi(1)} = \frac{1}{2} z_{\varphi(p)} + \frac{1}{2} z_{\varphi(q)}$. Since φ is arbitrary we can write $z_i = \sum_{j=1}^{n} a_{ij} z_j$, $i = 1, \cdots, n$, where for each i, there are precisely two values of j for which $a_{ij} = \frac{1}{2}$, and otherwise $a_{ij} = 0$. Let $A = (a_{ij})$, $z = (z_1, \cdots, z_n)$ so that

$$Az = z . \tag{4}$$

The matrix A may or may not be reducible but in any event there exists an n-square permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ * & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ * & \cdots & * & A_m \end{bmatrix} \tag{5}$$

and moreover each of the square matrices appearing along the main diagonal in (5) is irreducible or 1-square. Now suppose A_1 is k-square. Since each row of A (and hence of $P^T AP$) has precisely two nonzero entries in it, it follows that $k \geqq 2$. From (4) we have

$$P^T APx = x \tag{6}$$

where $x = P^T z$. Let $y = (x_1, \cdots, x_k)$ and we see that (5) and (6) imply that

$$(A_1 - I_k)y = 0 . \tag{7}$$

Since F has characteristic 0 it contains the rationals and A_1 can be regarded as a nonnegative, irreducible, row-stochastic matrix. By the Perron-Frobenius theorem [3, p. 124] we can immediately conclude that 1 is a simple eigenvalue of A_1 and hence the nullity of $A_1 - I_k$ over the rationals is $k - 1$. But the nullity is unchanged by regarding $A_1 - I_k$ as a matrix over any extension field of F. Now $e = (1, \cdots, 1)$ is in the null space of $A_1 - I_k$ and hence any vector y satisfying (7) must be a multiple of e. Since $k \geqq 2$ we conclude that at least two of the y_i are the same and hence that at least two of the z_i are the same. This contradiction completes the proof.

The preceding result has an immediate corollary. We let $\nu(\mathfrak{A})$ denote the cardinality of a set $\mathfrak{A}$.

COROLLARY. *Let $\mathfrak{A}$ be the set of all distinct non-zero differences of the form $z_i - z_j$, $i \neq j$. Then $\nu(\mathfrak{A})$ is even and at least $2n - 2$.*

Proof. According to the preceding argument there exists a permutation $\varphi \in S_n$ such that the $2n - 2$ differences $\pm(z_{\varphi(1)} - z_{\varphi(i)})$, $i = 2, \cdots, n$, are distinct. We can assume without loss of generality that φ is the identity. Let

$$\alpha = \{z_1 - z_i, i = 2, \cdots, n\} ,$$
$$\beta = \{z_i - z_1, i = 2, \cdots, n\} ,$$

$\nu(\alpha) = \nu(\beta) = n - 1$. If $\mathfrak{A} = \alpha \cup \beta$ then we are finished. So assume that there exist integers $i_1, j_1, 1 < i_1 \leqq n, 1 < j_1 \leqq n, i_1 \neq j_1$ such that $z_{i_1} - z_{j_1} \notin \alpha \cup \beta$. But then clearly $z_{j_1} - z_{i_1} \notin \alpha \cup \beta$. For if $z_{j_1} - z_{i_1} \in \alpha$, say, then $z_{j_1} - z_{i_1} = z_1 - z_t$ and hence $z_{i_1} - z_{j_1} = z_t - z_1$ in contradiction to the assumption that $z_{i_1} - z_{j_i} \notin \alpha \cup \beta$. Now set

$$\alpha_1 = \alpha \cup \{z_{i_1} - z_{j_1}\} , \qquad \beta_1 = \beta \cup \{z_{j_1} - z_{i_1}\} .$$

Clearly $\nu(\alpha_1 \cup \beta_1) = \nu(\alpha \cup \beta) + 2$ and if $\alpha_1 \cup \beta_1 \neq \mathfrak{A}$ we can repeat the preceding argument with α_1 and β_1 replacing α and β to obtain α_2 and β_2 such that $\nu(\alpha_2 \cup \beta_2) = \nu(\alpha_1 \cup \beta_1) + 2 = \nu(\alpha \cup \beta) + 4 = (2n - 2) + 4$. This procedure can obviously be continued until $\mathfrak{A}$ is exhausted.

To prove Theorem 1 we use a well known theorem of Roth [4]: if $(\lambda - \gamma_i)^p$ and $(\lambda - \gamma_j)^q$ are a pair of elementary divisors of the characteristic matrix of A then corresponding to these is a list of elementary divisors of the characteristic matrix of $A \otimes I_n - I_n \otimes A$:

$$(\lambda - (\gamma_i - \gamma_j))^t ,$$

where $t \leqq p + q - 1$. According to Theorem 2 there are at least $(2k - 2)$ distinct nonzero differences of the form $\pm(\gamma_{\varphi(1)} - \gamma_{\varphi(j)})$, $j = 2, \cdots, k$, and it is simply a matter of notational convenience to assume that these $2k - 2$ differences are $\pm(\gamma_1 - \gamma_i)$, $i = 2, \cdots k$. The highest degree elementary divisor involving the zero eigenvalue of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ is

$$\lambda^{2E-1} . \tag{8}$$

By the corollary, the set $\mathfrak{A}$ of all nonzero distinct eigenvalues of T_A is of the form

$$\begin{aligned} \mathfrak{A} = &\{\pm(\gamma_1 - \gamma_i),\ i = 2, \cdots, k\} \\ &\cup \{\pm(\gamma_{i_t} - \gamma_{j_t}),\ t = 1, \cdots, p\} . \end{aligned}$$

Now suppose the highest degree elementary divisors of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ involving the nonzero distinct eigenvalues of T_A are:

$$\begin{aligned} &(\lambda - (\gamma_1 - \gamma_i))^{e_{r_i} + e_{s_i} - 1},\ (\lambda - (\gamma_i - \gamma_1))^{e_{r_i} + e_{s_i} - 1},\ i = 2, \cdots, k , \\ &(\lambda - (\gamma_{i_t} - \gamma_{j_t}))^{e_{m_t} + e_{q_t} - 1},\ (\lambda - (\gamma_{j_t} - \gamma_{i_t}))^{e_{m_t} + e_{q_t} - 1},\ t = 1, \cdots, p . \end{aligned}$$

Thus the degree of the minimal polynomial of T_A is

$$d = 2E - 1 + 2 \sum_{i=2}^{k} (e_{r_i} + e_{s_i} - 1) + 2 \sum_{t=1}^{p} (e_{m_t} + e_{q_t} - 1) , \tag{9}$$

an odd integer. Observe that

$$d \geqq (2E - 1) + 2 \sum_{i=2}^{k} (e_1 + e_i - 1) + 2 \sum_{t=1}^{p} (e_{i_t} + e_{j_t} - 1) ,$$

and hence

$$\begin{aligned} d &\geqq (2E - 1) + 2 \sum_{i=2}^{k} e_i + 2(k - 1)(e_1 - 1) \\ &= 2E - 1 + 2(m - e_1) + 2(k - 1)(e_1 - 1) \\ &\geqq 2[m + E + (k - 2)e - k] + 1 . \end{aligned}$$

This completes the proof of Theorem 1.

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