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ERGODIC AUTOMORPHISMS AND AFFINE TRANSFORMATIONS OF LOCALLY COMPACT GROUPS

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It has been conjectured that if a locally compact group G has a *continuous automorphism* which is ergodic with respect to Haar measure then G must be compact. This is true when G is commutative or connected. In this paper further results in support of this conjecture are presented. *In particular, it is shown that the problem can be reduced* to the consideration of compactly generated, totally disconnected, locally compact groups without compact, open, normal subgroups and that the conjecture holds for many automorphisms of a certain class of such groups. Finally, the structure of locally compact groups which admit ergodic affine transformations is investigated.

The question of the existence of ergodic automorphisms on noncompact groups was first raised by P. Halmos [5, p. 29]. The commutative case was studied in [10] and [15] and the connected case in [8] and [14]. Theorem 1.1 below has been announced without proof in [16], and some of the results in § 2 have been obtained independently and by somewhat different methods by R. Sato [11], [12] and by N. Aoki and Y. Ito [1].

1. **Automorphisms.** Let G be a locally compact group and T an ergodic (whence bi-continuous and measure preserving, as shown in [10]) automorphism of G. Thus, if λ denotes a left Haar measure on G, $\lambda(A) = 0$ or $\lambda(A^c) = 0$ for any measurable subset A of G such that $T(A) = A$. Let G_0 denote the identity component of G.

THEOREM 1.1. *If G/G_0 is compact then G must be compact. Thus if there exists a noncompact group with an ergodic automorphism then there exists a noncompact totally disconnected one.*

Proof. If G/G_0 is compact there is a unique maximal normal compact subgroup N of G such that G/N is a Lie group [9, p. 175], [6, § XV. 3]. Since N must be invariant under T, there is induced an ergodic automorphism $\tilde{T}$ of G/N (cf. [8]). Since $(G/N)_0$ is open and invariant under $\tilde{T}$, G/N is in fact connected, whence compact. Thus G is compact.

If G is a noncompact group with ergodic automorphism T, then by the above the totally disconnected group G/G_0 must be noncompact, and it possesses the ergodic automorphism $\underset{\sim}{\tilde{T}}$ induced by T as above.

For the remainder of this section G will be assumed to be non-discrete and totally disconnected with ergodic automorphism T.

THEOREM 1.2. *Suppose that G satisfies one of the following conditions*:

(i) *Every compact subset of G is contained in a compact subgroup of G.*

(ii) *G has a compact, open, normal subgroup.*

Then G must be compact.

Proof. To prove that (i) implies compactness of G it clearly suffices to prove that G must be compactly generated. Let H be a compact, open subgroup of G, and for each n let K_n be the group generated by $H \cup T(H) \cup \cdots \cup T^n(H)$. Then $K_1 \subset K_2 \subset \cdots$ and $T(K_n) \subset K_{n+1}$, $n = 1, 2, \cdots$. Set $K = \bigcup_{n=1}^{\infty} K_n$, so that K and $T(K)$ are open subgroups of G with $T(K) \subset K$.

Suppose $T(K) \neq K$, and let $S = K \backslash T(K)$. Note that

$$T^m(S) \cap T^n(S) = \emptyset$$

if $m \neq n$, and $\lambda(S) > 0$. Choose $P \subset S$ with $\lambda(P) > 0$ and $\lambda(S \backslash P) > 0$ and set $Q = \bigcup_{n=-\infty}^{\infty} T^n(P)$. Then $T(Q) = Q$ and $Q \cap (S \backslash P) = \emptyset$, contradicting the ergodicity of T. Thus $T(K) = K$, so $K = G$ since K is closed. Hence $T^{-1}(H) \subset K_n$ for some n, whence $T^{-1}(K_n) \subset K_n$. Repeating the argument above we obtain $T^{-1}(K_n) = K_n$, so $G = K_n$ is compactly generated.

If (ii) holds, let H be a compact, open, normal subgroup of G and let the K_n be as defined above. Then each $K_n = H \cdot T(H) \cdots T^n(H)$ is compact, so the argument above shows G is compact.

COROLLARY 1.3. *If G is the union of an increasing sequence of compact open subgroups, then G is compact.*

COROLLARY 1.4. *If G is nilpotent, then it must be compact.*

Proof. The proof is by induction on the length of the ascending central series of G, $\{e\} \subset Z(G) \subset Z^2(G) \subset \cdots \subset Z^n(G) = G$. If $n = 1$, then G is abelian, whence compact. In general, since $Z(G)$ is invariant under T, $G/Z(G)$ has an ergodic automorphism (cf. Theorem 1.1) and $Z^{n-1}(G/Z(G)) = G/Z(G)$. If we assume that this implies $G/Z(G)$ is compact, we may invoke the main theorem of [4] to conclude that (ii) of Theorem 1.2 holds, whence G is compact, and the induction is complete.

COROLLARY 1.5. *If G is maximally almost periodic* (i.e., has sufficiently many finite-dimensional continuous unitary representations to separate points) *then it must be compact.*

Proof. Since we have shown G is compactly generated, it is known that G, being totally disconnected, must satisfy (ii) of Theorem 1.2 [7, Corollary XII. 3, pp. 58–59].

Miscellaneous Remarks.

(i) G must be unimodular. This follows either from a calculation which shows that the modular function, which is continuous, is invariant under T or from the fact that the modular function is a homomorphism which must equal one on the finite collection of compact subgroups which generate G as in the proof of Theorem 1.2.

(ii) In testing the validity of our conjecture it suffices to consider only metrizable groups. Indeed, if G and T are as above let H be a compact normal subgroup of G such that G/H is metrizable. Let $K = \bigcap_{n=-\infty}^{\infty} T^n(H)$. Then $T(K) = K$, so G/K has an ergodic automorphism and is metrizable.

(iii) If there exists an ergodic inner automorphism, $T(x) = a^{-1}xa$ for some $a \in G$, then G must be noncompact and the subgroup of G generated by a must be discrete (and infinite). For otherwise there is a compact subgroup K of G containing a. If H is a compact open subgroup of G, then clearly $G = KHK$ and is thus compact. But then every neighborhood of the identity in G contains a normal open subgroup, which is impossible.

We conclude this section by showing, in Theorem 1.8, that the conjecture in question holds for many automorphisms of a class of totally disconnected groups which do not, in general, satisfy (i) or (ii) of Theorem 1.2. We are grateful to A. Borel for his suggestions regarding the proof of Theorem 1.8.

LEMMA 1.6. *Let H be a closed normal subgroup of G such that G/H is compact and H is the union of an increasing sequence of compact open subgroups. Then G is also the union of an increasing sequence of compact open subgroups.*

Proof. Let $H_1 \subset H_2 \subset \cdots \subset H$ as in our hypothesis, and choose a symmetric compact subset K of G such that $G = HK$. Since $K^2 = K \cdot K$ is compact, it is contained in H_nK for some n. Then

$$K^3 \subset H_nK^2 \subset H_n^2K = H_nK ,$$

and so on. Thus, denoting by $[K]$ the subgroup generated by K,

$$[K] = \bigcup_{m=1}^{\infty} K^m \subset H_nK ,$$

so $[K]^-$ is compact. Hence we may assume K is a group.

Let G_n be the closed subgroup generated by H_n and K, $n = 1, 2, \cdots$. Then $G = \bigcup_{n=1}^{\infty} G_n$, so by the Baire category theorem G_n is open for all sufficiently large n. To show each G_n is compact, notice that, given n, we can find a positive integer m such that $h^x = xhx^{-1} \in H_m$ for all $h \in H_n$ and $x \in K$. It follows that $[H_n, K] \subset H_m K$, for every element of $[H_n, K]$ is of the form

$$h_1 x_1 \cdots h_r x_r = h_1 h_2^{x_1} h_3^{x_1 x_2} \cdots h_r^{x_1 \cdots x_r} x_1 \cdots x_r \; (h_i \in H_n,\; x_i \in K)$$

and hence lies in $H_m K$.

COROLLARY 1.7. *If G has a normal series consisting of closed subgroups whose successive quotients are all unions of increasing sequences of compact open subgroups, then G is also the union of such an increasing sequence.*

Proof. We have $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$, and we apply induction on n. The case $n = 1$ leaves nothing to prove; in general G_{n-1} is normal in G. Let $\pi: G \rightarrow G/G_{n-1}$ be the canonical map, and write $G/G_{n-1} = H_1 \cup H_2 \cup \cdots$, the union of an increasing sequence of compact open subgroups. By the induction hypothesis and Lemma 1.6 we may write $\pi^{-1}(H_m)$ as the union of such an increasing sequence:

$$\pi^{-1}(H_m) = \bigcup_{k=1}^{\infty} H_{mk} , \qquad m = 1, 2, \cdots ,$$

and by choosing subsequences we may assume $H_{m+1,k} \supset H_{mk}$ for all k and m. Let us now choose a sequence $\{G_m^0\}$ of compact open subgroups of G as follows. Let $G_1^0 = H_{11}$, and suppose $G_1^0, \cdots, G_m^0$ have been chosen. Let $G_{m+1}^0 = H_{m+1\, k}$, k being so large that $G_{m+1}^0 \supset G_m^0 \cup H_{mm}$. Then the G_m^0 are increasing, and it is clear that

$$\bigcup_{m=1}^{\infty} G_m^0 \supset \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} H_{mk} = G .$$

Let k be a nondiscrete, locally compact, totally disconnected field. If G is a connected (in the Zariski topology) linear algebraic group defined over k (k-group) [2], [3], $G(k)$ will denote the rational elements of G over k and $\mathrm{Aut}_k G$ the group of automorphisms of G defined over k. The restriction to $G(k)$ of any $T \in Aut_k G$, denoted also by T, is a topological automorphism of $G(k)$ when $G(k)$ is given its locally compact topology obtained by realizing it as a subgroup of $GL(n, k)$ for some n. Let R be the radical of G and U the unipotent radical. Recall that R [U] is the unique maximal connected, solvable [unipotent], Zariski-closed, normal subgroup of G. If k is of characteristic zero, then U (whence also R) is always defined over k. However if k has positive characteristic this need not be the case.

THEOREM 1.8. *Let G be a connected linear algebraic group defined over the nondiscrete, locally compact, totally disconnected field k such that U is defined over k, and let $T \in \mathrm{Aut}_k G$. If T is ergodic on $G(k)$, then $G(k)$ is compact and solvable. In fact, if char $k = 0$ then G is an anisotropic torus.*

Proof. Since R/U is the (Zariski-) connected component of the identity of the center of G/U [3, Proposition 11.21], R is defined over k. Let $G' = G/R$ and $\pi: G \to G/R$ the canonical map. G' is connected, defined over k, and semi-simple. And $T(R) = R$, so T induces an automorphism T' of G', defined over k, whose restriction to $\pi(G(k)) = G(k)/R(k)$ is an ergodic automorphism. Since G' is semi-simple, Aut G' is a k-group, and we have $(\mathrm{Aut}\ G')(k) = \mathrm{Aut}_k G'$. Furthermore the semi-direct product $H = G' \cdot (\mathrm{Aut}\ G')$ is also a k-group, so $H(k) = G'(k) \cdot (\mathrm{Aut}_k G')$ may be realized as a subgroup of $GL(n, k)$ for some n. Let $H(k)$ be so realized. Then T' is given on $G(k)$ by conjugation by some element of $GL(n, k)$. Given $\lambda \in k$, consider the continuous mapping $\varphi_\lambda: \pi(G(k)) \to k$ given by

$$\varphi_\lambda(x) = \det(x - \lambda e).$$

φ_λ is invariant under T', whence constant on $\pi(G(k))$. Thus

$$\varphi_\lambda(x) = \varphi_\lambda(e) = (1 - \lambda)^n , \qquad \lambda \in k,\ x \in \pi(G(k)) .$$

Hence every element of $\pi(Gk))$ has as characteristic polynomial $(1 - \lambda)^n$, i.e., $\pi(G(k))$ consists of unipotent matrices.

Recall that $G(k)$ has a natural structure as a k-analytic variety of analytic dimension equal to the dimension of G [13, Appendix III]. Since G is connected $G(k)$ cannot be contained in any proper analytic subset of G. Thus $G(k)$ is Zariski-dense in G, so $\pi(G(k))$ is dense in G'. Thus G' is both unipotent and semi-simple, whence trivial.

Thus $G = R$ is solvable, so G/U is a torus [3, Theorem 10.6], whence abelian. As above T induces an ergodic automorphism of $G(k)/U(k)$, so$G\ (k)/U(k)$ is compact. Moreover, U is trigonalizable over k [3, Corollary 15.5], and k is the union of an increasing sequence of compact open subgroups. By Lemma 1.6 and Corollary 1.7 we conclude that $U(k)$ is the union of such a sequence, and hence so is $G(k)$. Corollary 1.3 then implies $G(k)$ is compact.

If k is of characteristic zero, whence perfect, and if U were nontrivial, then $U(k)$ would have a normal series of closed subgroups whose successive quotients are isomorphic to the additive group of k. But $G(k)$ being compact, this is impossible, and it follows that G is an anisotropic torus.

REMARK. Using some deeper results from the theory of algebraic groups, Theorem 1.8 can be generalized in several directions. These generalizations will be presented elsewhere.

2. Affine transformations. Let G be a locally compact group and T an affine transformation of G. That is, $T(x) = a\tau(x)$, $x \in G$, for some continuous automorphism τ of G and $a \in G$. As in § 1, we are interested in structural results about G implied by the assumption that T is ergodic. We no longer expect that G must be compact; for example, $T(x) = x + 1$ is ergodic on Z.

The proofs of our first two lemmas are analogous to those of the corresponding facts about automorphisms found in [8] and [10].

LEMMA 2.1. *If T is ergodic then it must be bicontinuous and measure preserving.*

LEMMA 2.2. *Let H be a closed normal subgroup of G such that $\tau(H) = H$, and let $\tilde{\tau}$ denote the induced automorphism of G/H. If T is ergodic on G then the affine transformation $\tilde{T}(\bar{x}) = \bar{a}\tilde{\tau}(\bar{x})$ is ergodic on G/H.*

LEMMA 2.3. *Let G be discrete and T ergodic on G. Then G is finitely generated.*

Proof. A computation shows that for $n > 0$,

(1) $T^n(e) = a\tau(a) \cdots \tau^{n-1}(a)$,

(2) $T^{-n}(e) = (\tau^{-n}(a)\tau^{-(n-1)}(a) \cdots \tau^{-1}(a))^{-1} = (\tau^{-n}(T^n(e)))^{-1}$.

There exists $p \in Z$ such that $T^p(e) = \tau(a)$. Clearly $p = 0$ (i.e., T an automorphism) is impossible, while $p = 1$ gives $T(a^n) = a^{n+1}$ and this is the example cited above.

Suppose $p > 1$. Then from (1) we have

$$\tau^{p-1}(a) = (a\tau(a) \cdots \tau^{p-2}(a))^{-1}\tau(a) ,$$

and it follows that

$$T^n(e) \in [a, \tau(a), \cdots, \tau^{p-2}(a)] = H$$

for all $n > 0$. And applying τ^{-1} to (1) gives $\tau^{-1}(a) \in H$ and hence $T^{-n}(e) \in H$, $n > 0$. Thus $G =$ orbit of $e = H$ is finitely generated.

Now assume $p < 0$. Then by (2) we have

$$T^n(e) \in [a, \tau^{-1}(a), \cdots, \tau^p(a)] = K, \qquad n > 0 ,$$

while applying τ^{-1} to (2) shows $\tau^{p-1}(a) \in K$ and hence $T^{-n}(e) \in K$, $n > 0$.

Again G is finitely generated.

THEOREM 2.4. *If G has an ergodic affine transformation, then G is compactly generated. If G has a nontrivial, compact, open, normal subgroup, then G is compact.*

Proof. If G is discrete, then it is finitely generated by Lemma 2.3, so we assume G is nondiscrete. Assume first that G has a compact open subgroup H. Let K_n be the group generated by

$$\{a, \tau(a), \cdots, \tau^{n-1}(a)\} \cup H \cup \cdots \cup \tau^n(H), \qquad n = 1, 2, \cdots .$$

The proof that G is compactly generated now proceeds like that of Theorem 1.2 (here $T(K)$ is an open left coset), except that we now choose n so large that $\{\tau^{-1}(a)\} \cup \tau^{-1}(H) \subset K_n$.

Now let G be any locally compact group with identity component G_0, and let $\pi: G \to G/G_0$ be the natural map. By Lemma 2.2 and the argument above there is a compact generating set C for G/G_0. If U is a compact neighborhood of the identity in G such that $\pi(U) \supset C$, then $U \cap G_0$ generates G_0, and it is easy to see that U generates G.

Suppose the H chosen above is normal and nontrivial. Let

$$H_n = \tau^{-n}(H) \cdots H \cdots \tau^n(H) , \qquad n = 1, 2, \cdots ,$$

and set $H^* = \bigcup_{n=1}^{\infty} H_n$, so that $\tau(H^*) = H^*$. Then by Lemma 2.2 G/H^* is discrete and has an ergodic affine transformation. If G/H^* is infinite then $T^m(H^*) \cap T^n(H^*) = \emptyset$ if $m \neq n$. But this clearly implies $H = H^* = \{0\}$, as in the proof of Theorem 1.2, a contradiction.

Thus G/H^* is finite, and we have $T^m(a) \in H^*$ for some m. Let $A = \{e, a, \cdots, \tau^{m-1}(a)\}$. Replacing H by H_n for sufficiently large n, we may assume that $T^m(a) \in H$ and AH is a subgroup of G (see proof of Lemma 1.6). The argument above then shows

$$G = K_n = AH\tau(H) \cdots \tau^n(H)$$

for some n, so G is compact.

COROLLARY 2.5. *If G is maximally almost periodic and has an ergodic affine transformation, then G is discrete or compact.*

Proof. See Corollary 1.5.

LEMMA 2.6. *R^n has no ergodic affine transformations.*

Proof. (cf. [5, p. 28]) Induction on n. It is easy to see no affine transformation on R is ergodic. Assume R^{n-1} has no ergodic affine transformations, and let $T(x) = \tau(x) + a$ be affine on R^n. Let $\pi: R^n \to R^{n+1}$ be given by $\pi(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 1)$, and let S be the linear transformation of R^{n+1} given by the matrix

$$\begin{pmatrix} \tau & a \\ 0 & 1 \end{pmatrix},$$

so that $S \circ \pi = \pi \circ T$. By Lemma 2.1 we have $\det S = \det \tau = \pm 1$. Suppose first that τ is unipotent. Then τ has an eigenvector in R^n, so we may apply Lemma 2.2 and the induction assumption to conclude that T is not ergodic.

If τ is not unipotent we shall construct a nonconstant continuous function on the hyperplane $R^n \times \{1\}$ which is invariant under S, proving that T is not ergodic. Consider S^*, which is given by the matrix

$$\begin{pmatrix} \tau^* & 0 \\ a & 1 \end{pmatrix}$$

on the dual space $(R^{n+1})^*$. Since τ, whence τ^*, is not unipotent, neither is S^*. Thus the complexification S_c^* of S^* has an eigenvalue $\lambda \neq 1$. The annihilator of C^n in $(C^{n+1})^*$ consists of eigenvectors of S_c^* with eigenvalue 1. It follows, then, that any eigenvector for λ assumes arbitrarily large values on $R^n \times \{1\} \subset C^{n+1}$. Indeed, if $z(x)$ is such an eigenvector, then z does not annihilate C^n, so being complex linear, it cannot annihilate R^n. Thus z assumes arbitrarily large values on R^n, whence on $R^n \times \{1\}$. Let $z_1, \cdots, z_k$ denote eigenvectors corresponding to the distinct eigenvalues of S_c^* whose respective multiplicities are $n_1, \cdots, n_k$, and set

$$f(x) = |z_1(x)^{n_1} \cdots z_k(x)^{n_k}| .$$

Then $f \circ S = f$ and f is not constant.

LEMMA 2.7. *Let T be an ergodic affine transformation of Z^n. Then $n = 1$ and $T(x) = x \pm 1$.*

Proof. It is easy to see that if $n = 1$ then $T(x) = x \pm 1$. The rest of the proof is by an induction similar to that of Lemma 2.6: τ is given by a matrix with integer coefficients and determinant ± 1. If τ is not unipotent the argument above shows that T is not ergodic. (Note that f assumes arbitrarily large values on $Z^n \times \{1\}$.) On the other hand, if τ is unipotent then it has an eigenvector over Q, whence also over Z. Let $x_0 \in Z^n$ be an eigenvector and $H = Qx_0 \cap Z^n$. By Lemma 2.2 the induced affine transformation of $Z^n/H \cong Z^{n-1}$ is ergodic. The lemma now follows by induction once we settle the case $n = 2$. But in this case $Z^2/H \cong Z$ and the induced affine map is a translation. Whence the orbit of 0 under T is a proper subset of Z^2, so T is not ergodic.

THEOREM 2.8. *Let G be abelian and have an ergodic affine transformation. Then $G = Z$ or G is compact.*

Proof. If G is discrete and infinite, then by Lemma 2.3 and Theorem 2.4 G is finitely generated and torsion free, so Lemma 2.7 implies $G = Z$.

Suppose G is nondiscrete, and let G_0 denote its identity component. Then G/G_0 must be finite or nondiscrete. For G/G_0 discrete and infinite implies $G/G_0 = Z$, and this leads to a contradiction as in the proof of Theorem 2.4. If G/G_0 is nondiscrete, then by Theorem 2.4 G/G_0 is compact. Thus in general, $G = R^n \times H$ with H compact. Since H is invariant under any automorphism of G, it follows from Lemmas 2.2 and 2.6 that $n = 0$ and $G = H$ is compact.

COROLLARY 2.9. *If G is nilpotent and has an ergodic affine transformation, then $G = Z$ or G is compact.*

Proof. The proof is analogous to that of Corollary 1.4.

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