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FIXED POINT THEOREMS FOR NONLINEAR NONEXPANSIVE AND GENERALIZED CONTRACTION MAPPINGS

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Let X be a reflexive Banach space, H a closed convex subset of X, and let K be a nonempty, bounded, closed and convex subset of H which possesses normal structure. If $T: K \to H$ is nonexpansive and if $T: \partial_H K \to K$ where $\partial_H K$ denotes the boundary of K relative to H, then T has a fixed point in K. This result generalizes an earlier theorem of the author, and a more recent theorem of F. E. Browder. An analogue is given for generalized contraction mappings in conjugate spaces.

1. **Introduction.** In [13] we proved that if K is a nonempty, bounded, closed and convex subset of a reflexive Banach space, and if K possesses "normal structure" (defined below), then every nonexpansive mapping T of K into K has a fixed point. This result, also proved independently by F. E. Browder [4] and D. Göhde [11] (in uniformly convex spaces), initiated rather extensive study of fixed point theory for nonlinear nonexpansive operators in Banach spaces, including applications to the study of nonlinear equations of evolution by Browder [5] and to certain nonlinear functional equations (see Browder and Petryshyn [8], Kolomý [16], Srinivasacharyulu [21]).

In this paper we modify the approach of [13] to treat the following problem: Given closed and convex subsets K and H of a Banach space X such that $K \cap H \neq \emptyset$ and an operator $T: K \to X$ such that (i) $T: K \cap H \to H$ and (ii) $T: \partial_H K \to K$ (where $\partial_H K$ denotes the relative boundary of $K \cap H$ in H), when does T have a fixed point? This kind of problem has been of particular interest in the case where the operator T is completely continuous, H is the positive cone of X, and the fixed points of T correspond to positive solutions of a differential equation (for example, see [17]). A standard approach is to use the technique of "radial projection" to associate with T an operator B which is also completely continuous, has the same fixed points as T, and maps the intersection of H with the ball $K: \|x\| \leqq R$ into itself, thus permitting application of the classical Schauder Theorem [19]. Such an approach, however, is not suitable for our purposes because we consider mappings of nonexpansive type. Since radial projection is in general *not* nonexpansive (see [9]), the associated operator B need not be nonexpansive and one cannot obtain a fixed point by direct application of the theorem of [13].

Before stating our results we establish relevant notation and definitions.

A mapping T of a subset K of a Banach space X into X is called *nonexpansive* if $\| Tx - Ty \| \leqq \| x - y \|$ for all $x, y \in K$.

For a subset S of a Banach space X, the symbol $\delta(S)$ denotes the *diameter* of S—i.e.,

$$\delta(S) = \sup \{\| x - y \|;\ x, y \in S\} .$$

The notation $U(z; r)$ is used to denote the spherical neighborhood of z of radius $r > 0$:

$$U(z; r) = \{x \in X\colon \| z - x \| < r\} .$$

Similarly,

$$\bar{U}(z; r) = \{x \in X\colon \| z - x \| \leqq r\} .$$

The concept of normal structure, due to Brodskii and Milman [3], plays a key role in our approach. A bounded convex set K in a Banach space X is said to have *normal structure* if for each convex subset S of K which contains more than one point, there is a point $x \in S$ which is a *nondiametral point of* S (i.e., $\sup \{\| x - y \| \colon y \in S\} < \delta(S)$). *Compact convex sets possess normal structure* ([3], [10, Lemma 1]) *as do all bounded convex subsets of uniformly convex spaces.* (For a comparison of normal structure and uniform convexity, see Belluce-Kirk-Steiner [2]. The concept has also been studied by Gossez and Lami Dozo [12].)

We wish to thank the referee for his suggestions, particularly for pointing out the corollary to Theorem 3.1.

2. A fixed point theorem for nonexpansive mappings.

For H and K subsets of X, we use the symbol $\partial_H K$ to denote the boundary of K relative to H: Thus, letting $H - K$ denote the points of H which are not in K, if K is closed,

$$\partial_H K = \{z \in K\colon\ U(z;\ r) \cap (H - K) \neq \emptyset \text{ for each } r > 0\} .$$

THEOREM 2.1. *Let X be a reflexive Banach space, H a closed convex set in X, and K a nonempty, bounded, closed, convex subset of H which possess normal structure. If $T\colon K \to H$ is nonexpansive, and if $T\colon \partial_H K \to K$, then T has a fixed point in K.*

The above theorem immediately reduces to our theorem of [13] upon taking $H = K$. A more interesting consequence of this theorem arises from taking $H = X$:

COROLLARY. *Let K be a bounded closed convex subset of a reflexive Banach space X and suppose K possesses normal structure. If $T: K \to X$ is nonexpansive, and if T maps the boundary of K into K, then T has a fixed point in K.*

Browder first obtained the above result [6, Theorem 3] for K a bounded closed convex set in a uniformly convex space with the additional assumption that T is defined on an open convex set $G \supset K$ with $\operatorname{dist}(K, X - G) > 0$. Subsequently Browder [7] and Nussbaum [18] have removed this assumption (in a uniformly convex setting) while proving more general results, a fact which is significant because in general one may not enlarge the domain of nonexpansive mapping [20].

Proof of Theorem 2.1. Let $\mathscr{T}$ be the family of all closed convex subsets of H such that for $F \in \mathscr{T}$, $F \cap K \neq \emptyset$ and $T: F \cap K \to F$. Since $H \in \mathscr{T}$, $\mathscr{T} \neq \emptyset$. Let $\{F_\alpha\}$ be a descending chain of sets of $\mathscr{T}$, and let $F = \bigcap_\alpha F_\alpha$. Note that $F \cap K$ is nonempty, since each of the sets $F_\alpha \cap K$ is a nonempty weakly compact subset of X. Also, since $T: F_\alpha \cap K \to F_\alpha$ for each α, clearly $T: F \cap K \to F$. Since F is closed and convex, $F \in \mathscr{T}$, and therefore by Zorn's Lemma, $\mathscr{T}$ has a minimal element.

Letting F be such a minimal element of $\mathscr{T}$, first *note that we may sssume* $\partial_F K \neq \emptyset$, for otherwise $F \subset K$ and $T: F \cap K \to F$ would imply $T: F \to F$. The existence of a fixed point would then follow from the theorem of [13].

Now we assume $\delta(F \cap K) > 0$ and obtain a contradiction. Let $\delta = \delta(F \cap K)$. Since K possesses normal structure, there exists a point $c \in F \cap K$ such that

$$\sup\{\|c - z\|: z \in F \cap K\} = r < \delta .$$

Let

$$C = \{x \in X: F \cap K \subset \bar{U}(x; r)\} .$$

It is easily seen that C is closed and convex and, since $c \in F \cap C$, $(F \cap C) \cap K \neq \emptyset$. Also there exist points $x, y \in F \cap K$ such that $\|x - y\| > r$. Such points cannot be elements of C and therefore $F \cap C$ is a proper subset of F. We complete the proof by showing $F \cap C \in \mathscr{T}$. Since we have already seen that $(F \cap C) \cap K \neq \emptyset$, we need only show that $T: (F \cap C) \cap K \to F \cap C$.

Suppose $z \in (F \cap C) \cap K$. Let

$$W = \bar{U}(Tz; r) \cap F .$$

If $W \in \mathscr{T}$, then since $W \subset F$ and F is minimal, $W = F$. This implies

$$F \cap K \subset F \subset \bar{U}(Tz; r) ,$$

and hence $Tz \in C$. Since $T: F \cap K \to F$, this in turn yields $Tz \in F \cap C$. Therefore $T: (F \cap C) \cap K \to F \cap C$ if $W \in \mathscr{T}$ for every $z \in (F \cap C) \cap K$. We complete the proof by showing this to be the case.

First suppose $x \in W \cap K$. Then $x \in F \cap K$ so $\| x - z \| \leqq r$ (because $z \in C$). Hence $\| Tx - Tz \| \leqq r$ and $Tx \in \bar{U}(Tz; r)$. But $x \in W \cap K$ also implies $x \in F \cap K$ and hence $Tx \in F$. Therefore $Tx \in \bar{U}(Tz; r) \cap F = W$, i.e., $T: W \cap K \to W$.

Finally, since $\partial_F K \neq \emptyset$, it follows that $W \cap K \neq \emptyset$. To see this, note that if $y \in \partial_F K$ then $y \in F \cap K$ and $\| y - z \| \leqq r$, which implies $\| Ty - Tz \| \leqq r$ and therefore $Ty \in W$. But also $\partial_F K \subset \partial_H K$ implies $Ty \in K$; hence $Ty \in W \cap K$ and $W \cap K \neq \emptyset$.

This completes the proof that $F \cap C \in \mathscr{T}$, contradicting the assumption $\delta(F \cap C) > 0$. Thus $\delta(F \cap C) = 0$ and $F \cap C$ consists of a single point which, because $T: \partial_F K \to K$, is fixed under T.

3. Generalized contraction mappings. In this section we give an analogue of Theorem 2.1 for the class of generalized contraction mappings studied in [14, 15]. With X a Banach space, and $K \subset X$, a mapping $T: K \to X$ is called a *generalized contraction mapping* if for each $x \in K$ there is a number $\alpha(x) < 1$ such that

$$\| Tx - Ty \| \leqq \alpha(x) \| x - y \| \qquad \text{for each } y \in K .$$

It was noted in Belluce-Kirk [1] that mappings of this type provide an example of a class of mappings with "diminishing orbital diameters"; thus fixed point theorems proved in [1] apply to this class of mappings. In [15] it is shown that if A is a bounded open convex subset of X and if $F: A \to X$ is continuously Fréchet differentiable on A, then F is a generalized contraction mapping on A if and only if for each $x_0 \in A$ the norm of the Fréchet derivative F'_{x_0} of F at x_0 is less than one. It is also shown that if K is a w^*-compact convex subset of a conjugate Banach space X and if $T: K \to K$ is a generalized contraction mapping, then T has a fixed point in K. This result may be generalized as follows:

THEOREM 3.1. *Let X be a conjugate Banach space, H a convex w^*-closed subset of X, and K a nonempty convex w^*-compact subset of H. If $T: K \to H$ is a generalized contraction mapping on K, and if $T: \partial_H K \to K$, then T has a fixed point in K.*

Proof. As in the proof of Theorem 2.1, obtain a w^*-compact convex set F minimal with respect to the properties $F \cap K \neq \emptyset$ and

$T\colon F \cap K \to F$. As before, it may be assumed that $\partial_F(K) \neq \emptyset$ (otherwise $F \subset K$ and existence of a fixed point follows from Theorem 1.1 of [15]).

The argument parallels that of Theorem 2.1 upon obtaining a point $c \in F \cap K$ such that

$$\sup\{\|c - z\|\colon z \in F \cap K\} < \delta \,. \tag{1}$$

Such a point can be obtained by letting $x \in \partial_F K$, noting that $Tx \in F \cap K$, and using the procedure of the proof of Theorem 1.1 of [15] to show that Tx has the property specified for c in (1). Specifically, one can show that if $\delta = \delta(F \cap K) > 0$ then for the number $\alpha(x) < 1$ associated with T by definition, one has

$$\bar{U}(Tx; \alpha(x)\delta) \cap F \in \mathscr{F}$$

which implies

$$\sup\{\|Tx - z\|\colon z \in F \cap K\} \leqq \alpha(x)\delta = r < \delta \,.$$

Then letting $Tx = c$, define the set C as in Theorem 2.1 and observe that

$$C = \bigcap_{x \in F \cap K} \bar{U}(x; r) \,.$$

Thus C is w^*-compact and convex. This and the fact that the set W defined later in the argument is also w^*-compact and convex, enables one to complete the proof precisely as in Theorem 2.1. We omit the details.

COROLLARY. *If X is a conjugate Banach space and H is a closed convex subset of X of which every intersection with a w^*-compact set is w^*-compact* (e.g. $H = X$), *and if $T\colon H \to H$ is a generalized contraction mapping on H, then T has a fixed point in H.*

Proof. Let $x \in H$ and let

$$K = H \cap \bar{U}\left(x; \frac{\|x - Tx\|}{1 - \alpha(x)}\right) .$$

Then $T\colon \partial_H K \to K$ and by Theorem 3.1, T has a fixed point in H.

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