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CLASSIFYING SPECIAL OPERATORS BY MEANS OF SUBSETS ASSOCIATED WITH THE NUMERICAL RANGE

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Let A be a continuous linear operator on a complex Hilbert space X, with inner product $<,>$ and associated norm $\|\ \|$. For each complex number z let $M_z(A) = \{x: \langle Ax, x\rangle = z\|x\|^2\}$. The following classifications of special operators are obtained: (i) A is a scalar multiple of an isometry if and only if $AM_z(A) \subset M_z(A)$ for each complex z; (ii) A is a nonzero scalar multiple of a unitary operator if and only if $AM_z(A) = M_z(A)$ for each complex z; and (iii) A is normal if and only if for each complex z $\{x \mid Ax \in M_z(A)\} = \{x \mid A^*x \in M_z(A)\}$.

1. Introduction. The sets, $M_z(A)$, are closely associated with the numerical range of A: $W(A) = \{\langle Ax, x\rangle: \|x\| = 1\}$. These sets were introduced in [1] and used to characterize the elements of $W(A)$ as follows:

THEOREM A. *If $z \in W(A)$, then*
(i) *z is an extreme point of $W(A)$ if and only if $M_z(A)$ is linear,*
(ii) *if z is a nonextreme boundary point of $W(A)$, then*

$$\gamma M_z(A) = \cup\{M_w(A): w \in L\}$$

where L is the line of support for $W(A)$ passing through z,
(iii) *if $W(A)$ is a convex body, then z is an interior point of $W(A)$ if and only if $\gamma M_z(A) = X$.*

It was also shown in [1, Theorem 2] that $\cap$ {maximal linear subspaces of $M_z(A)$} plays a special role in determining the normal eigenvalues of A.

With the aforementioned evidence concerning the sets $M_z(A)$ in mind, it seemed natural to ask whether these sets behave in a particular fashion if A has special characteristics or whether the action of A on these sets determines special properties of A. Obviously A is Hermitian if and only if $M_z(A) = M_{z^*}(A)$ for all complex z. The first question which came to mind was: when is it the case that each of the sets $M_z(A)$ is invariant under A. The techniques developed to answer this question in Theorem 1 led to the other theorems in this paper.

The following elementary facts can be noted about the sets, $M_z(A)$. 1. Each set $M_z(A)$ is homogeneous and 2. either $M_z(A) \cap M_w(A) = \{0\}$ or $M_z(A) = M_w(A)$.

2. Notation and terminology. The notation and terminology used in this paper are the same as that found in [1] with the following additions. f is a *bilinear functional* on a complex vector space X if and only if $f\colon X \times X \rightarrow \{\text{complex numbers}\}$, f is linear in the first variable and conjugate linear in the second variable.

Throughout the paper A is a continuous linear operator on a complex Hilbert space X; A is an *isometry* if $A^*A = I$; A is *unitary* if $A^*A = AA^* = I$; A is *normal* if $AA^* = A^*A$; and A is *hyponormal* if $AA^* \leqq A^*A$. ker A denotes the null space of A: $\{x\colon Ax = 0\}$.

3. Classification theorems. The following lemma plays a fundamental part in the proofs of Theorems 1–4.

LEMMA 1. *If f, g, h and k are bilinear functionals on a complex vector space X, satisfying*

$$f(x, x)g(x, x) = h(x, x)k(x, x) \text{ for all } x \text{ in } X, \text{ then} \tag{1}$$

$$f(x, y)g(x, y) = h(x, y)k(x, y) \text{ for all } x \text{ and } y \text{ in } X. \tag{2}$$

Indication of proof. Let $x, y \in X$ and let z be an arbitrary complex number. By substituting $y + zx$ for x in equation (1) and equating coefficients, one arrives at equation (2) by means of the coefficients of z^2.

THEOREM 1. *A is a scalar multiple of an isometry if and only if $AM_z(A) \subset M_z(A)$ for each complex z.*

Proof. $M_z(A)$ is invariant under A for each complex z if and only if

$$\langle A^2x, Ax\rangle \| x \|^2 = \langle Ax, x\rangle \| Ax \|^2 \text{ for all } x \text{ in } X . \tag{3}$$

Obviously if A is a scalar multiple of an isometry, then equation (3) holds for all x in X. Thus we assume that equation (3) holds for all x in X and by Lemma 1 have

$$\langle A^2x, Ay\rangle \langle x, y\rangle = \langle Ax, y\rangle \langle Ax, Ay\rangle \text{ for all } x \text{ and } y \text{ in } X . \tag{4}$$

It now follows that $\{x\}^\perp \subset \{Ax\}^\perp \cup \{A^*Ax\}^\perp$. Moreover with x and y interchanged in (4) we see that $\{x\}^\perp \subset \{A^*x\}^\perp \cup \{A^*Ax\}^\perp$. Since $\{y\}^\perp$

is linear, we have either $\{x\}^{\perp} \subset \{A^*Ax\}^{\perp}$ or $\{x\}^{\perp} \subset \{Ax\}^{\perp} \cap \{A^*x\}^{\perp}$. Either case implies that there exists a scalar r_x such that $A^*Ax = (r_x)x$. This is sufficient to imply that A is a scalar multiple of an isometry.

If A is a nonunitary isometry, the only complex z in $W(A)$ for which $AM_z(A) = M_z(A)$ are the extreme points of $W(A)$. To prove this we make use of results from [2] and [3] which assert that in this case $\sigma(A) = \overline{W(A)} = \{z\colon |z| \leqq 1\}$. Thus the elements of $W(A)$ are either extreme points z with $|z| = 1$ or interior points. If z is an extreme point of $W(A)$, then since A is hyponormal,

$$M_z(A) = \{x\colon Ax = zx \quad \text{and } A^*x = z^*x\}$$

by [4] and thus $M_z(A) = AM_z(A) = A^*M_z(A)$. Conversely if $M_z(A) = AM_z(A)$, then $\gamma M_z(A) = A(\gamma M_z(A))$. By Theorem A, (iii) if z is an interior point of $W(A)$, then $X = AX$, implying that A is invertible and hence unitary. Therefore if $M_z(A) = AM_z(A)$ and $z \in W(A)$, then z is an extreme point of $W(A)$.

THEOREM 2. *A^* is a scalar multiple of an isometry if and only if $A^*M_z(A) \subset M_z(A)$ for each complex z.*

Proof. Apply Theorem 1 to A^* and note that $M_z(A^*) = M_{z^*}(A)$ for each complex z.

THEOREM 3. *A is a nonzero scalar multiple of a unitary operator if and only if $AM_z(A) = M_z(A)$ for each complex z.*

Proof. By Theorems 1 and 2 A is a scalar multiple of a unitary operator if and only if $AM_z(A) \subset M_z(A)$ and $A^*M_z(A) \subset M_z(A)$ for each complex z. Thus if A is nonzero, this is equivalent to $AM_z(A) \subset M_z(A)$ and $M_z(A) \subset AM_z(A)$.

The proof of Theorem 4 which classifies normal operators in terms of the sets $M_z(A)$ appears to depend upon the following lemma.

LEMMA 2. *If A and E are operators on X such that ker $A \subset$ ker E and for each x in X either*

(i) $||Ax|| = ||Ex||$

or

(ii) *there exists a real number r_x such that*

$$A^*Ax = (r_x)E^*Ex\ ,$$

then A^*A is a scalar multiple of E^*E.

Proof. Assume that $A^*Ax = aE^*Ex$ and $A^*Ay = bE^*Ey$ where E^*Ex and E^*Ey are linearly independent. Let t be real, $0 < t < 1$. Either $||A(tx + (1 - t)y|| = ||E(tx + (1 - t)y||$ or there exists a real number c such that $A^*A(tx + (1 - t)y) = cE^*E(tx + (1 - t)y$. In this last case since $0 < t < 1$ and E^*Ex and E^*Ey are linearly independent, we have $a = c = b$. Thus if $a \neq b$, then

$$||A(tx + (1 - t)y)|| = ||E(tx + (1 - t)y)||$$

for all t, $0 < t < 1$. Letting t approach 1 and 0, we have $||Ax|| = ||Ex||$ and $||Ay|| = ||Ey||$. Therefore $|a| = |b| = 1$ and since $E^*Ex \neq 0$ and $E^*Ey \neq 0$, necessarily $a = b = 1$. Thus we must have $a = b$ if E^*Ex and E^*Ey are linearly independent.

Secondly if E^*Ex and E^*Ey are linearly dependent and $A^*Ax = aE^*Ex$ and $A^*Ay = bE^*Ey$, then it follows from the hypothesis $\ker A \subset \ker E$ that a and b can be chosen to be the same real number.

The arguments in the two preceding paragraphs show that there exists a real number r such that if $x \in X$, then either $A^*Ax = rE^*Ex$ or $||Ax|| = ||Ex||$. Thus either $||Ax|| \leqq ||Ex||$ for all x in X or $||Ax|| \geqq ||Ex||$ for all x in X. In either case $\{x: ||Ax|| = ||Ex||Ex||\}$ is linear by Theorem A, (i). proving that X is the union of the two linear subspaces:

$$\{x: A^*Ax = rE^*Ex\} \quad \text{and} \quad \{x: ||Ax|| = ||Ex||\} .$$

Therefore either $A^*A = rE^*E$ or $A^*A = E^*E$.

THEOREM 4. *A is normal if and only if for each complex z*

$$\{x \mid Ax \in M_z(A)\} = \{x \mid A^*x \in M_z(A)\} .$$

Proof. If A is normal it follows that $Ax \in M_z(A)$ if and only if $A^*x \in M_z(A)$. Assume now that this condition holds. Then

$$\langle A^2x, Ax\rangle ||A^*x||^2 = \langle AA^*x, A^*x\rangle ||Ax||^2 \text{ for all } x \text{ in } X \tag{5}$$

and

$$\ker A = \ker A^* . \tag{6}$$

This last assertion can be proven as follows: $x \in \ker A \leftrightarrow Ax \in M_z(A)$ for all complex $z \leftrightarrow A^*x \in M_z(A)$ for all complex $z \leftrightarrow x \in \ker A^*$.

Using the same techniques as in the proof of Theorem 1, we show that if $x \in X$, either their exists a number b such that $AA^*x = bA^*Ax$ or there exist numbers c and d such that $AA^{*2}x = cAA^*x$ and $A^*A^2x = dA^*Ax$. These last two equations combined with (5) and (6) imply that either $Ax = A^*x = 0$ or $c = d^*$. They also imply that $A^{*^2}x =$

cA^*x and $A^2x = dAx$. Again using (6), we have $AA^*x = cAx$ and $A^*Ax = dA^*x$. Thus if $Ax \neq 0$, $\|A^*x\|^2 = c\langle Ax, x\rangle = d^*\langle x, A^*x\rangle = \|Ax\|^2$. Therefore A and A^* satisfy the hypotheses of Lemma 2 and there exists a real number r such that $AA^* = rA^*A$. This is sufficient to imply that A is normal.

COROLLARY 5. *Let A be an invertible operator on X. The following statements are equivalent:*

(i) *A is normal,*

(ii) *$A^{-1}M_z(A) = A^{*-1}M_z(A)$ for each complex z,*

(iii) *$A^{-1}M_z(A^*A^{-1}) = A^{*-1}M_z(A^*A^{-1})$ for each complex z.*

Proof. The equivalence of (i) and (ii) is a restatement of Theorem 4 for the case in which A is invertible. The equivalence of (i) and (iii) is obtained by applying Theorem 3 to the operator A^*A^{-1}.

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