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AXIOMATIC CONVEXITY THEORY AND RELATIONSHIPS BETWEEN THE CARATHÉODORY, HELLY, AND RADON NUMBERS

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An axiomatic setting for the theory of convexity is provided by taking an arbitrary set X and constructing a family $\mathscr{C}$ of subsets of X which is closed under intersections. The pair consisting of any ordered vector space and its family of convex subsets thus become the prototype for all such pairs $(X, \mathscr{C})$. In this connection, Levi proved that a Radon number r for $\mathscr{C}$ implies a Helly number $h \leqq r-1$; it is shown in this paper that exactly one additional relationship among the Carathéodory, Helly, and Radon numbers is true, namely, that if $\mathscr{C}$ has Carathéodory number c and Helly number h then $\mathscr{C}$ has Radon number $r \leqq ch+1$. Further, characterizations of (finite) Caratheodory, Helly, and Radon numbers are obtained in terms of separation properties, from which emerges a new proof of Levi's theorem, and finally, axiomatic foundations for convexity in euclidean space are discussed, resulting in a theorem of the type proved by Dvoretzky.

1. Preliminary definitions. A family of subsets of a space X which is closed under intersection yields a weak type of closure, or hull, operator on the power set of X, producing concepts which may be readily applied to convexity and topology alike. Our main interest is, however, convexity theory and the abstraction of certain classical concepts from that area. (See in this regard the papers by Danzer, Grünbaum and Klee [1], Hammer [5, 6, 7], Eckhoff [3], Ellis [4], Koenen [8], and Levi [9].) We shall, therefore, introduce the following terminology: A family $\mathscr{C}$ of subsets of a set X is termed a *convexity structure* for X, with the pair $(X, \mathscr{C})$ being called a *convexity space*, whenever the following two conditions hold:

(a) $\varnothing$ and X belong to $\mathscr{C}$;

(b) $\bigcap \mathscr{F} \in \mathscr{C}$ for each subfamily $\mathscr{F} \subset \mathscr{C}$.

$\mathscr{C}$ is designated T_1 iff the further condition

(c) $\{x\} \in \mathscr{C}$ for each $x \in X$

holds, and a subfamily $\mathscr{B}$ of $\mathscr{C}$ is called a *basis* of $\mathscr{C}$ iff each member of $\mathscr{C}$ is obtainable as an intersection of members of $\mathscr{B}$.

The hull operator generated by a convexity structure $\mathscr{C}$, defined in the usual manner by the relation

$$\mathscr{C}(S) = \bigcap\{C \in \mathscr{C} : C \supset S\}\,, \qquad S \subset X\,,$$

enjoys certain properties identical to those of the closure operator in topology, among which are: (i) $S \subset \mathscr{C}(S)$ for each $S \subset X$; (ii) if $S_1 \subset S_2$ the $\mathscr{C}(S_1) \subset \mathscr{C}(S_2)$; (iii) $\mathscr{C}(\mathscr{C}(S)) = \mathscr{C}(S)$; and, (iv) $S \in \mathscr{C}$ iff $\mathscr{C}(S) = S$. The set $\mathscr{C}(S)$ will be termed the *$\mathscr{C}$-hull* of S, and a set will be called *$\mathscr{C}$-convex* iff $\mathscr{C}(S) = S$. If S is finite and consists of the points $x_1, \cdots, x_k$ we shall write simply $\mathscr{C}(x_1, \cdots, x_k)$ for its $\mathscr{C}$-hull.

An important concept in ordinary convexity theory is that of the "cone" or "join" of a point over a set. We may extend this concept to our general setting by defining the *$\mathscr{C}$-join of x and S* to be the set

$$x_{\mathscr{C}}S \equiv \bigcup_{s \in S} \mathscr{C}(x, s) \text{ .}$$

A useful condition involving this concept is the following (for $x \in X$ and $S \subset X$):

(d) $\mathscr{C}(x \cup S) \subset x_{\mathscr{C}}\mathscr{C}(S)$.

Since the reverse inclusion can be easily proved, (d) is equivalent to the condition

(d′) $\mathscr{C}(x \cup S) = x_{\mathscr{C}}\mathscr{C}(S)$.

It is interesting that (d) is also equivalent to assuming that the $\mathscr{C}$-join and $\mathscr{C}$-hull operators *commute* at $x \in X$, that is, for each $S \subset X$,

(d″) $\mathscr{C}(x_{\mathscr{C}}S) = x_{\mathscr{C}}\mathscr{C}(S)$.

This may be seen by simply verifying the relation $\mathscr{C}(x \cup S) = \mathscr{C}(x_{\mathscr{C}}S)$. A convexity structure satisfying either of the equivalent conditions (d), (d′), or (d″) will be called *join-hull commutative at x*, and if $\mathscr{C}$ is join-hull commutative for each $x \in X$ it will be said to be *join-hull commutative*. Further, we say that $\mathscr{C}$ is *finitely* join-hull commutative if (d), (d′), or (d″) holds for each $x \in X$ and for each finite subset $S \subset X$. (Condition (d) was introduced for finite subsets by Ellis [4].)

The next property is the direct analogue of the classical Carathéodory theorem on convex hulls in a vector space over an ordered field, and will reveal a relationship between join-hull commutativity and finite join-hull commutativity:

(e) $\mathscr{C}(S) = \bigcup\{\mathscr{C}(T) \colon T \subset S, |T| < \infty\}$ for each $S \subset X$.

After Hammer [6], a convexity structure $\mathscr{C}$ having property (e) will be termed *domain finite*.

The following two theorems will illustrate the application of these properties.

THEOREM 1. *If $\mathscr{C}$ is a convexity structure for X which is domain finite, then finite join-hull commutativity implies join-hull com-*

mutativity.

Proof. It suffices to show that for $x \in X$ and $S \subset X$, $\mathscr{C}(x \cup S) \subset x_{\mathscr{C}}\mathscr{C}(S)$. Let $y \in \mathscr{C}(x \cup S)$; then there exists a finite set $T \subset S$ such that $y \in \mathscr{C}(x \cup T)$ and

$$\mathscr{C}(x \cup T) \subset x_{\mathscr{C}}\mathscr{C}(T) \subset x_{\mathscr{C}}\mathscr{C}(S).$$

Hence, $y \in x_{\mathscr{C}}\mathscr{C}(S)$.

THEOREM 2. *If $\mathscr{C}$ is a convexity structure for X which is join-hull commutative and domain finite, then a set $C \subset X$ is $\mathscr{C}$-convex iff $\mathscr{C}(x, y) \subset C$ for each $x \in C$, $y \in C$.*

Proof. Suppose C is $\mathscr{C}$-convex. Then if $x \in C$ and $y \in C$, $\mathscr{C}(x, y) \subset \mathscr{C}(C) = C$. Conversely, suppose for each $x \in C$ and $y \in C$, $\mathscr{C}(x, y) \subset C$; we observe that the hypothesis implies by join-hull commutativity that for any finite set $T \subset C$, $\mathscr{C}(T) \subset C$. It follows immediately that $\mathscr{C}(C) = C$, for, by domain finiteness,

$$\mathscr{C}(C) = \bigcup\{\mathscr{C}(T) \colon T \subset C, |T| < \infty\} \subset C .$$

Finally, a convexity structure $\mathscr{C}$ is said to have *Carathéodory number c* iff c is the smallest positive integer for which it is true that the $\mathscr{C}$-hull of any set $S \subset X$ is the union of the $\mathscr{C}$-hulls of those subsets of S of cardinality $\leqq c$. Further, a convexity structure has *Helly number h* and *Radon number r* iff h and r are the smallest positive integers for which it is true that, respectively, a finite subfamily $\mathscr{F}$ of sets in $\mathscr{C}$ has nonempty intersection provided each h members of $\mathscr{F}$ has nonempty intersection, and any set S with $|S| \geqq r$ has a *Radon partition*, that is, may be partitioned into two nonempty subsets (S_1, S_2) such that $\mathscr{C}(S_1) \cap \mathscr{C}(S_2) \neq \varnothing$.

These definitions imply that in general $c \geqq 1$, $h \geqq 1$, and $r \geqq 2$, and that for any T_1 convexity space having at least 3 points, $c \geqq 1$, $h \geqq 2$, and $r \geqq 3$. The least value for c in either case is attained by taking $\mathscr{C}$ to be the largest possible convexity structure for X (consisting of the power set of X), and the least values for h and r are obtained when $\mathscr{C}$ is the smallest possible $[T_1]$ convexity structure (consisting of $\varnothing$, [the singleton subsets of X], and X). If $\mathscr{C}$ is the family of convex sets in euclidean space E^d of dimension d the classical theorems of Carathéodory, Helly, and Radon imply that in this case $\mathscr{C}$ has $c = h = d + 1$ and $r = d + 2$. It is easy to construct examples to show that convexity structures can have a variety of possible Carathéodory, Helly, and Radon numbers, but in general there will be certain restrictions.

2. Interrelationships between the numbers *c*, *h*, and *r*. Levi's theorem [9] shows that in any convexity space $(X, \mathscr{C})$ if $\mathscr{C}$ has Radon number r then $\mathscr{C}$ has Helly number $h \leqq r - 1$. To show that no other possible relationships between the numbers c, h, and r exist (taken singly) we cite the following examples (also discussed in part using different definitions by M. Breen in a related unpublished paper communicated to the authors by W. R. Hare and J. W. Kenelly): Take $X = E^2$ and consider Hammer's example of the convexity structure generated by X and sets of the form $C = H_1 \cup H_2 \cup (L_1 \sim L_2)$, where L_1 and L_2 are any two perpendicular lines and H_1 and H_2 are open half planes determined by them. As proved in [5], this convexity structure has Carathéodory number 7 but no finite Helly or Radon number. The example consisting of $X = E^d$ and all closed convex sets in X provides a convexity structure which has Helly number $d + 1$, Radon number $d + 2$, but no finite Carathéodory number (since no point on the boundary of an open convex set S is contained in the closed, convex hull—and thus $\mathscr{C}$-hull—of any finite subset of S).

The above two examples show that among the Carathéodory, Helly, and Radon numbers c, h, and r, the existence of c does not imply that of either h or r, and neither the existence of h nor r implies that of c. It remains to show that the existence of h does not imply that of r. To that end, consider the following example.

EXAMPLE 1. In the sequence space $X = E^\infty = \{(x_1, \cdots, x_i, \cdots): x_i \in R\}$ (R = reals), let $\mathscr{C}$ consist of $\varnothing$, X, and the collection of all closed and bounded rectangular hypersolids with faces orthogonal to the coordinate axes, explicitly defined as $C \equiv \bigcap_{i \in N} C_i$ (N = positive integers), where, for each i,

$$C_i = \{x: a_i \leqq x_i \leqq b_i\}\ , \qquad a_i \leqq b_i\ ,$$

x_i denoting the i^{th} coordinate of x. (This is Eckhoff's product $\prod_{i=1}^\infty (X_i, \mathscr{C}_i)$ with $X_i = E^1$ and $\mathscr{C}_i$ = family of closed intervals; see [3] and a related paper by Reay [10].)

It is clear that if $(X, \mathscr{C})$ is the convexity space defined in Example 1 and $S \subset X$,

$$\mathscr{C}(S) = \{x: a_i \leqq x_i \leqq b_i,\ i \in N\}\ ,$$

where

$$a_i = \inf_{x \in S} x_i\ , \qquad b_i = \sup_{x \in S} x_i\ .$$

From reasoning of a similar nature it follows that $\mathscr{C}$ has Helly number 2. But we show that $\mathscr{C}$ has no (finite) Radon number. Let

$k \in N$, $k \geqq 2$, and with $n = \sum_{j=1}^{k-1} \binom{k}{j} = 2^k - 2$, construct the $k \times n$ matrix M_k of zeroes and ones as follows: The first $\binom{k}{l}$ columns define the characteristic functions of all one-element subsets of $\{1, \cdots, k\}$, the next $\binom{k}{2}$ columns define the characteristic functions of all two-element subsets, and in general, the $\binom{k}{l}$ columns $1 + \sum_{j=1}^{l-1} \binom{k}{j}$ through $\sum_{j=1}^{l} \binom{k}{j}$ define the characteristic functions of all l-element subsets of $\{1, \cdots, k\}$, $1 \leqq l \leqq k-1$. Note that for $k = 5$ (in which case $n = 30$), this process yields the 5×30 matrix

$$M_5 = \begin{bmatrix} 10000 & 1111000000 & 1111110000 & 11110 \\ 01000 & 1000111000 & 1110001110 & 11101 \\ 00100 & 0100100110 & 1001101101 & 11011 \\ 00010 & 0010010101 & 0101011011 & 10111 \\ 00001 & 0001001011 & 0010110111 & 01111 \end{bmatrix}.$$

Now let S be the k-element subset of X defined by taking those points whose first n coordinates are given by the *rows* of M_k and whose remaining coordinates are each zero. If (S_1, S_2) is any nontrivial partitioning of S then $1 \leqq |S_1| \leqq k-1$ and there is an integer i such that the i^{th} coordinate of each member of S_1 is 1 and the i^{th} coordinate of each member of S_2 is 0. Thus, the i^{th} coordinate of each member of $\mathscr{C}(S_1)$ is 1, and the i-th coordinate of each member of $\mathscr{C}(S_2)$ is 0, from which it follows that $\mathscr{C}(S_1) \cap \mathscr{C}(S_2) = \varnothing$. Therefore, no k-element subset of X has a Radon partition, and since k was arbitary, $\mathscr{C}$ has no radon number.

The preceding examples show that Levi's theorem is the only one possible if we assume the finiteness of exactly one of the numbers c, h, or r; however a possible implication arises by considering *pairs* of numbers c, h, or, r and this is answered conclusively by the following theorem.

THEOREM 3. *If $\mathscr{C}$ is a convexity structure for X which has Carathéodory number c and Helly number h, then $\mathscr{C}$ possesses a Radon number $r \leqq ch + 1$.*

Proof. Let S be a $(ch+1)$-element subset of X, and define $\mathscr{F}$ to be the subsets of S having at least $ch + 1 - c$ clements. By a simple counting argument it follows that each h members of the family $\mathscr{G} = \{\mathscr{C}(F): F \in \mathscr{F}\} \subset \mathscr{C}$ have nonempty intersection: Let $G_i = \mathscr{C}(F_i)$ for each $F_i \in \mathscr{F}$, $i = 1, \cdots, h$, and consider $|\bigcup_i (S \sim G_i)| \leqq \bigcup_i |S \sim G_i| \leqq \bigcup_i |S \sim F_i| \leqq h[(ch+1) - (ch + 1 - c)] \leqq ch < |S|$.

Since $\mathscr{C}$ has Helly number h there is a point x belonging to $\bigcap \mathscr{G}$, and because $S \in \mathscr{F}$ and $\mathscr{C}(S) \in \mathscr{G}$ we have $x \in \mathscr{C}(S)$. But $\mathscr{C}$ has Carathéodory number c and therefore we can find a subset T of S of cardinality c or less such that $x \in \mathscr{C}(T)$. Since $S \sim T$ has cardinality at least $ch + 1 - c$ then $\mathscr{C}(S \sim T) \in \mathscr{G}$ and $x \in \mathscr{C}(S \sim T)$. Hence, the pair $(T, S \sim T)$ is a Radon partition of S, proving that $\mathscr{C}$ has Radon number $r \leqq ch + 1$.

COROLLARY 1. *In a convexity space having finite Carathéodory number c, the existence of a Helly number h and a Radon number r are equivalent, and the corresponding numbers satisfy the inequality*

$$h + 1 \leqq r \leqq ch + 1 .$$

The following example due to Eckhoff [3] sheds further light on the general behavior of the Carathéodory, Helly, and Radon numbers.

EXAMPLE 2. With $X = E^d$ and for a given integer $k > 0$ let $\mathscr{C}$ consist of all the convex subsets of X and all finite subsets $S \subset X$ such that $|S| \leqq k$.

Since $\mathscr{C}$ contains the usual convex subsets of E^d it is clear that $\mathscr{C}$ has Carathéodory number $c = d + 1$. Eckhoff proves that if $2 \leqq d \leqq k + 1$, then $\mathscr{C}$ has Radon number $2k + 2$, and by Levi's theorem $\mathscr{C}$ has finite Helly number h. By our Theorem 3,

$$2k + 2 = r \leqq h(d + 1) + 1 .$$

Thus, by allowing $k \to \infty$ we have a class of convexity structures in which the Carathéodory number is a constant (as small as 3) while both the Helly and Radon numbers take on arbitrarily large values.

3. A characterization of the numbers c, h, and r by separation properties. Since separation theorems bear prominently on problems in convexity, it is of interest to know how they may be related to the Carathéodory, Radon, and Helly properties in a more general setting. If two members H_1 and H_2 of a convexity structure $\mathscr{C}$ for X partition X they are called *complementary* $\mathscr{C}$*-half-spaces*. If S_1, S_2 are respectively contained by a complementary pair H_1, H_2 of $\mathscr{C}$-half-spaces, then S_1 and S_2 are said to be $\mathscr{C}$*-separated*.

The existence of $\mathscr{C}$-half-spaces and the possibility of separating disjoint members of $\mathscr{C}$ in general is a problem discussed by Ellis in [4], where he introduces a property which, together with join-hull commutativity and domain finiteness, will guarantee such separation. For our purposes, let us say that a convexity structure $\mathscr{C}$ has the *separation property* if it satisfies the axiom

(f) Each two disjoint members of $\mathscr{C}$ may be $\mathscr{C}$-separated.

Following Hammer, a closely related idea is the following: H' is called a *$\mathscr{C}$-semispace* iff it is a member of $\mathscr{C}$ which is maximal with respect to being disjoint from some other member of $\mathscr{C}$. An application of Zorn's lemma shows that for each two disjoint members C_1 and C_2 of $\mathscr{C}$ there exists a $\mathscr{C}$-semispace containing C_1 and disjoint from C_2, provided $\mathscr{C}$ is closed under unions of chains of its members (it can be proved that such is the case if $\mathscr{C}$ is domain finite). Thus, mere existence of $\mathscr{C}$-semispaces in $\mathscr{C}$ is no problem, but the complement of a $\mathscr{C}$-semispace may not be a member of $\mathscr{C}$. It turns out that an alternate way to handle the separation problem is to assume the following property, which can shown to be equivalent to (f) above in domain finite convexity structures:

(f′) The complement of each $\mathscr{C}$-semispace is $\mathscr{C}$-convex.

It is then clear that whenever (f) [or (f′)] is assumed in a domain finite convexity structure $\mathscr{C}$, with $\mathscr{S}$ and $\mathscr{H}$ denoting the $\mathscr{C}$-semispaces and $\mathscr{C}$-half-spaces of $\mathscr{C}$, $\mathscr{S} \subset \mathscr{H} \subset \mathscr{C}$; moreover, if $\mathscr{C}$ is T_1 then $\mathscr{S}$ and $\mathscr{H}$ are both bases for $\mathscr{C}$, with the members of $\mathscr{H}$ being generated by those of $\mathscr{S}$.

We now proceed to the characterization theorems mentioned earlier; the first two do not require domain finiteness.

THEOREM 4. *In any T_1 convexity structure $\mathscr{C}$ on X having the separation property, the following two conditions are equivalent:*

(i) *$\mathscr{C}$ has Helly number $h \leqq k$.*

(ii) *If S is a $(k+1)$-element subset of X, $k \geqq 2$, there exists $p \in X$ such that every $\mathscr{C}$-half-space containing at least k elements of S also contains p.*

Proof. (i) $\rightarrow$ (ii). If S is any $(k+1)$-element subset of X, form the sets $S_i = S \sim \{x_i\}$, where $x_i \in S$, and the family $\mathscr{F} = \{\mathscr{C}(S_i)\} \subset \mathscr{C}$, for $i = 1, \cdots, k+1$. Since each k members of $\mathscr{F}$ have nonempty intersection and $\mathscr{C}$ has Helly number $h \leqq k$, there is a point $p \in \bigcap \mathscr{F}$. Hence if H is a $\mathscr{C}$-half-space containing k elements of S, H contains one of the sets $\mathscr{C}(S_i)$ of $\mathscr{F}$, and hence p.

(ii) $\rightarrow$ (i). Suppose (ii) is satisfied for some $k \geqq 2$, and that (i) fails for that k. Then there is a smallest subfamily $\mathscr{F}$ of $\mathscr{C}$ for which it fails, with each k members thereof having nonempty intersection but $\bigcap \mathscr{F} = \varnothing$. Hence $|\mathscr{F}| \geqq k+1$, and it follows by standard arguments and the minimal property of $\mathscr{F}$ that $|\mathscr{F}| = k+1$. Let $C_1, \cdots, C_{k+1}$ be the members of $\mathscr{F}$, and for each $i = 1, \cdots, k+1$, choose $x_i \in \bigcap\{C_j : j \neq i\}$. Then $x_i \neq x_j$ for $i \neq j$, for otherwise $x_j \in \bigcap \mathscr{F}$. Hence let p be as in (ii) with $S = \{x_1, \cdots, x_{k+1}\}$ and suppose $p \notin C_l$ for some l. By the separation property there is a

$\mathscr{C}$-half-space H containing C_l but not p. But C_l, and therefore H, contains k members of S, so by (ii) H contains p, a contradiction.

THEOREM 5. *In any T_1 convexity structure $\mathscr{C}$ on X having the separation property, the following two conditions are equivalent:*

(i) *$\mathscr{C}$ has Radon number $r \leqq k$.*

(ii) *If S is a k-element subset of X, $k \geqq 3$, there exists a proper subset T of S and $p \in X$ such that every $\mathscr{C}$-half-space which contains either T or $S \sim T$ also contains p.*

Proof. Obvious since $\mathscr{C}(T) = \bigcap\{H \in \mathscr{H}: H \supset T)$ and $\mathscr{C}(S \sim T\} = \bigcap\{H \in \mathscr{H}: H \supset S \sim T\}$. Hence $\mathscr{C}(T) \cap \mathscr{C}(S \sim T)$ is nonempty iff every $\mathscr{C}$-half-space containing T meets every $\mathscr{C}$-half-space containing $S \sim T$ at some point p.

The Carathéodory number may also be formulated in terms of separation properties, but the additional property of domain finiteness is needed. Since the proof is a routine application of the definitions it will be omitted.

THEOREM 6. *Let $\mathscr{C}$ be a T_1 convexity structure for X which has the separation property and is domain finite. Then the following conditions are equivalent:*

(i) *$\mathscr{C}$ has Carathéodory number $c \leqq k$.*

(ii) *If S is a subset of X having at least $k+1$ elements and $p \in \mathscr{C}(S)$, there is a proper subset T of S such that every $\mathscr{C}$-half-space containing T also contains p.*

We now apply two of the above characterization theorems to obtain an alternate proof of Levi's theorem (in a less general setting).

THEOREM 7 (Levi). *Let $\mathscr{C}$ be a T_1 convexity structure for X which has the separation property. Then a Radon number r for $\mathscr{C}$ implies a Helly number $h \leqq r - 1$.*

Proof. Let S be an r-element subset of X. By property (ii) of Theorem 5 there exists $T \subset S$ and $p \in X$ such that every $\mathscr{C}$-half-space H containing either T or $S \sim T$ contains p. Let H be any $\mathscr{C}$-half-space containing $r-1$ points of S. Then it follows that H contains either T or $S \sim T$, and thus p, yielding property (ii) of Theorem 4. Hence, $\mathscr{C}$ has Helly number $h \leqq r - 1$.

REMARK. In view of the simple proof of Levi's theorem using separation properties one suspects there are additional relationships among the numbers c, h, and r in convexity spaces satisfying the above two properties (e) and (f).

The next result makes use of certain separation properties to show that under certain conditions the existence of a Radon number is a sufficient condition for the existence of a Carathéodory number. The property needed is known to be true for $X = E^d$ when $d = 1, 2, 3$, with $\mathscr{C}$ the usual convexity structure (for a discussion of related versions of generalizations of Radon's theorem, see [1, p. 118]).

(g) If S is a finite subset of X which has a Radon partition (S_1, S_2) and $p \in \mathscr{C}(S)$ but $p \notin \mathscr{C}(S_1) \cap \mathscr{C}(S_2)$, then S has a Radon partition (T_1, T_2) such that

$$\bigcap\{H \in \mathscr{H}: H \supset T_1, p \notin H\} \cap T_2 \neq \varnothing .$$

THEOREM 8. *Let $\mathscr{C}$ be a T_1 convexity structure for X having the separation property, domain finiteness, and the additional property* (g) *mentioned above. Then, if $\mathscr{C}$ has Radon number $r < \infty$, it has Carathéodory number $c \leqq r - 1$.*

Proof. Let G be a subset of X and $p \in \mathscr{C}(G)$; by domain finiteness there is a finite subset $S \subset G$ of minimal cardinality such that $p \in \mathscr{C}(S)$. If $|S| \geqq r$, then S has a Radon partition; so by hypothesis S has a Radon partition (T_1, T_2) for which there exists a point $q \in \bigcap\{H \in \mathscr{H}: H \supset T_1, p \notin H\} \cap T_2$. Let H be any $\mathscr{C}$-half-space containing $S \sim q$. Then $T_1 \subset H$ (since $q \in T_2$), and if $q \notin H$, from the choice of q it follows that $p \in H$; but if $q \in H$ then $S \subset H$ and again $p \in H$. Thus, $p \in \mathscr{C}(S \sim q)$, denying the minimal property of S. Therefore, $|S| < r$, and $\mathscr{C}$ has Carathéodory number $c \leqq r - 1$.

4. An axiomatic foundation for convexity in euclidean space. It is of fundamental interest to derive the convexity structure of euclidean space from an abstract convexity structure in the case $X = E^d$. This can be done by assuming the axioms below [in addition to the previous conditions (a) and (b)]. Since the system is independent it can be proved that this set of conditions is both necessary and sufficient. A more difficult problem arises if we do not assume a euclidean setting, or if the axioms themselves are stated intrinsically—that is, solely in terms of the members of $\mathscr{C}$. The more general problem of deriving necessary and sufficient conditions for a convexity space $(X, \mathscr{C})$ to be a vector space over an ordered field for which the members of $\mathscr{C}$ are the convex sets might be referred to as the *linearization problem* for convexity.

A *similarity transformation*, or *similitude*, is any transformation

$f: E^d \to E^d$ having the *dilation* ("contraction-expansion") *property*

$$e(f(x), f(y)) = \alpha e(x, y) , \qquad \alpha > 0 ,$$

where e denotes the euclidean metric. The geometric properties of such mappings are well known, and we do not state them here. A *direct* similitude is one for which the matrix representing f in the usual manner has positive determinant. Throughout the section, we assume that $X = E^d$, and the usual topology will be understood.

Axiom 1: $\mathscr{C}$ is closed under similitudes in X.

Axiom 2: $\mathscr{C}$ has a member of cardinality $\geqq 2$ which is bounded in X.

Axiom 3: For every finite set $S \subset X$, if $x \notin \mathrm{cl}\, \mathscr{C}(S)$ then $\mathscr{C}(x \cup S) \subset x_{\mathscr{C}}\mathscr{C}(S)$.[1]

Axiom 4: $\mathscr{C}$ has Helly number $h \leqq d + 1$.

Axiom 5: $\mathscr{C}$ is domain finite.

The first four axioms are needed to prove that the members of $\mathscr{C}$ are each convex. This constitutes a theorem of the type discovered by Dvoretzky [2], in that, the classical Helly property for E^d is used to derive convexity. The similarity ends there, for while Dvoretzky uses the assumption that $\mathscr{C}$ is closed under affine mappings (stronger than our Axiom 1) and the compactness of its members (stronger than our Axiom 2), he does not assume the very restrictive condition that $\mathscr{C}$ is closed under arbitrary intersection. The closure of $\mathscr{C}$ under intersections coupled with closure under affine mappings is quite strong indeed; for, it is not difficult to prove that the only additional assumption needed to obtain the convexity of each member of $\mathscr{C}$ is, for example, that the $\mathscr{C}$-hull of two points be connected in X or, alternatively, that the $\mathscr{C}$-hull of two points contain a third and be closed in X.

The following quite different independent set of axioms characterizing the usual convexity structure in E^d were given in Womble's dissertation, the first three of which imply that the members of $\mathscr{C}$ are convex:

Axiom 1′: $\mathscr{C}$ is closed under isometries in X.

Axiom 2′: If $x \in C$ and $C \in \mathscr{C}$ then $\{x\}$ and C can be weakly separated (in the ordinary sense) in the flat of least dimension containing x and C.

Axiom 3′: $\mathscr{C}$ is T_1

Axiom 4′: $\mathscr{C}$ is finitely join-hull commutative.

Axiom 5′: $\mathscr{C}$ is domain finite.

Axiom 6′: For $u \in \mathscr{C}(x, y)$ and $v \in \mathscr{C}(x, z)$ then $\mathscr{C}(u, z) \cap \mathscr{C}(v, y) \neq \varnothing$

[1] The restriction $x \notin \mathrm{cl}\, \mathscr{C}(S)$ is contrived to achieve independence.

(see Ellis [4]).

Now we prove the assertions made previously about Axioms 1–5. We shall let $L(x, y)$ denote the line (1-flat) determined by x and y (if $x \neq y$), and xy the join (ordinary convex hull) of x and y. Recall that any two corresponding points of a direct similitude may be specified in advance.

LEMMA 1. *If* $\mathscr{C}$ *satisfies Axioms* 1 *and* 2 *then for any two distinct points* x *and* y,

$$\mathscr{C}(x, y) \subset L(x, y) ,$$

and $\mathscr{C}$ *is* T_1.

Proof. First, note the two fundamental properties of any one-to-one mapping $f: X \rightarrow X$ having the property that for each $C \in \mathscr{C}$, $f[C]$ and $f^{-1}[C]$ are members of $\mathscr{C}$:

(i) If $S \subset X$ then $f[\mathscr{C}(S)] = \mathscr{C}(f[S])$.

(ii) If $f[S] \cup T \subset \mathscr{C}(S)$ then $f[T] \subset \mathscr{C}(S)$.

The first being routine, the second may be proved from the first by writing

$$f[T] \subset f[\mathscr{C}(S)] = \mathscr{C}(f[S]) \subset \mathscr{C}[\mathscr{C}(S)] = \mathscr{C}(S) .$$

In particular, (i) and (ii) hold if f is any similitude. Now suppose $C = \mathscr{C}(x, y)$ and $z \in C \sim L(x, y)$. Let f be a direct similitude which takes x to x, y to z, and leaves the plane of z and $L(x, y)$ invariant. With $z_{-1} = y$ and $z_0 = z$, define

$$z_n = f(z_{n-1}) \quad \text{and} \quad \angle(z_{n-1}, x, z_n) = \theta_n .$$

for $n = 0, 1, \cdots$. An inductive application of (ii) proves that $z_n \in C$ for all n. Note that $\theta_n = \theta_0$ for each n and that, therefore, $\angle(z_{-1}, x, z_n) = (n+1)\theta_0$ for all n such that $(n+1)\theta_0 \leqq \pi$. Thus, there is an integer n for which $\angle(z_{-1}, x, z_n) > \pi/2$. Set $u = z_n$ and let v be the reflection of u in the perpendicular bisector of xy. It follows that $v \in C$ and thus C contains a (perhaps degenerate) trapezoid (x, y, v, u) with $e(u, v) > e(x, y)$. If g is a direct similitude which maps x to u and y to v, again define $u_0 = u$, $v_0 = v$, and

$$u_{n+1} = g(u_n) , \qquad v_{n+1} = g(v_n)$$

for $n = 0, 1, \cdots$. It follows that $u_n \in C$ and $x_n \in C$ for all $n \geqq 0$ and $e(u_n, v_n) = \lambda^n e(u_0, v_0)$, where $\lambda = e(u, v)/e(x, y) > 1$. Thus, C is unbounded. But if B is the guaranteed set of Axiom 2 and h is a similitude which maps the two guaranteed points of B onto x and y then $h[B]$ is a member of $\mathscr{C}$ containing x and y and therefore

$$C = \mathscr{C}(x, y) \subset h[B] ,$$

contradicting the boundedness of B.

LEMMA 1'. *If $\mathscr{C}$ satisfies Axioms* 1 *and* 2 *then for any two distinct points x and y,*

$$\mathscr{C}(x, y) \subset xy .$$

Proof. Same proof as in Lemma 1, using a trapezoid of 0 height.

LEMMA 2. *If $\mathscr{C}$ satisfies Axioms* 1–3, *then for any finite set of points $x_1, \cdots, x_k$,*

$$\mathscr{C}(x_1, \cdots, x_k) \subset \operatorname{conv}(x_1, \cdots, x_k) .$$

Proof. The assertion is true for $k = 2$ by Lemma 1', so suppose it has been shown for any set of $k-1$ points, $k \geqq 3$. Let $x_1, \cdots, x_k$ be given and choose the notation so that x_1 is an extreme point of $\operatorname{conv}(x_1, \cdots, x_k)$ and put $S = \{x_2, \cdots, x_k\}$. Thus, $x_1 \notin \operatorname{cl}\operatorname{conv} S$ and by the induction hypothesis $\mathscr{C}(S) \subset \operatorname{conv} S$, so $x_1 \notin \operatorname{cl} \mathscr{C}(S)$. By Axiom 3,

$$\begin{aligned}\mathscr{C}(x_1, \cdots, x_k) = \mathscr{C}(x_1 \cup S) \subset (x_1)_{\mathscr{C}}\mathscr{C}(S) \subset \bigcup_{s \in \operatorname{conv} S} \operatorname{conv}(x_1, s) \\ = \operatorname{conv}(x_1, \cdots, x_k) .\end{aligned}$$

LEMMA 3. *If $\mathscr{C}$ satisfies Axioms* 1–4 *and $\mathscr{C}_k$ denotes the members of $\mathscr{C}$ lying in a flat of dimension k, then $\mathscr{C}_k$ has Helly number $k+1$, $1 \leqq k \leqq d$.*

Proof. Using induction on the deficiency $d - k$ of the flat, suppose the assertion has been proved for any collection $\mathscr{C}_k$ and let $C_1, \cdots, C_{k+1}$ be $k+1$ members of $\mathscr{C}_{k-1}$ contained in a $(k-1)$-flat F, each k of which have nonempty intersection. We first replace the C_i by $\mathscr{C}$-hulls P_i of finite point sets. For each i define $x_i \in \bigcap_{j=1, j\neq i}^{k+1} C_j$ and let $P_i = \mathscr{C}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1})$. It follows that $P_j \subset C_j$ for each j. Each k of $P_1, \cdots, P_{k+1}$ have nonempty intersection since $x_i \in P_j$ if $i \neq j$, and, therefore, $x_i \in \bigcap_{j=1, j\neq i}^{k+1} P_j$. Let $x \notin F$ and define the sets $C'_i = \mathscr{C}(x \cup P_i)$ for $1 \leqq i \leqq k+1$ and $C'_{k+2} = \mathscr{C}(x_1, \cdots, x_{k+1})$. Then by Lemma 2 the sets $C'_1, \cdots, C'_{k+2}$ lie in a k-flat and each $k+1$ have nonempty intersection. By the induction assumption there exists $p \in \bigcap_{i=1}^{k+2} C'_i$. For each i, one can use Axiom 3 and Lemma 2 to prove that

$$p \in C_i' \cap C_{k+2}' \subset F \cap x_{\mathscr{C}} P_i \subset F \cap \left(\bigcup_{s \in P_i} xs \right) = P_i \,.$$

The standard inductive argument may now be used to extend the property to any finite subcollection of $\mathscr{C}_{k-1}$, and hence $\mathscr{C}_{k-1}$ has Helly number k.

LEMMA 4. *If $\mathscr{C}$ satisfies Axioms 1–4, then for any finite set of points $x_1, \cdots, x_k$,*

$$\mathscr{C}(x_1, \cdots, x_k) = \operatorname{conv}(x_1, \cdots, x_k) \,.$$

Proof. Consider the case $k = 2$, and suppose $p \in xy \sim \mathscr{C}(x, y)$. The sets $C_1 = \mathscr{C}(x, p)$, $C_2 = \mathscr{C}(p, y)$, and $C_3 = \mathscr{C}(x, y)$ are subsets of $F = L(x, y)$ so that Lemma 3 applies, with $k = 2$. Each 2 of the C_i intersect, but $\bigcap_{i=1}^{3} C_i = \varnothing$ since $p \notin \mathscr{C}(x, y)$. The contradiction proves that $xy \subset \mathscr{C}(x, y)$ and establishes the result for $k = 2$. Induction may then be applied to finish the proof.

The obvious result of Lemma 4 is

THEOREM 9. *If $\mathscr{C}$ is a convexity structure for $X = E^d$ satisfying Axioms 1–4, then each member of $\mathscr{C}$ is convex.*

The independence of our axioms show that at this point $\mathscr{C}$ need not contain all the convex sets of X. But the use of Axiom 5 together with the result of Lemma 4 provides an easy proof of the final result.

THEOREM 10. *If $\mathscr{C}$ is a convexity structure for $X = E^d$ satisfying Axioms 1–5, then $\mathscr{C}$ is precisely the family of convex sets of X.*

Proof. Let $C \subset X$ be convex, and consider $x \in \mathscr{C}(C)$. By Axiom 5 there exist points $x_1, \cdots, x_k$ in C such that

$$x \in \mathscr{C}(x_1, \cdots, x_k) = \operatorname{conv}(x_1, \cdots, x_k) \subset \operatorname{conv} C = C \,.$$

Therefore $\mathscr{C}(C) \subset C$, implying $C = \mathscr{C}(C)$. Hence, $C \in \mathscr{C}$ and, in view of Theorem 9, the result is proved.

It is routine to show the independence of each of the above axioms, except for Axiom 2. For, in order to deny exactly one each of the Axioms 1, 3, 4, and 5, merely take $\mathscr{C}$ to be, respectively,

(i) $\varnothing$, X, and all convex subsets of diameter $\leqq 1$;

(ii) $\varnothing$, X, and all convex subsets of dimension $\leqq k$ for some fixed k, $2 \leqq k < d^2$;

[2] It is easy to find examples in (i) and (ii) for which there are nonconvex members of $\mathscr{C}$. This, together with Example 3, shows that no proper subsystem of Axioms 1–4 is sufficient to prove Theorem 9.

(iii) $\emptyset$, X, and all subsets of cardinality $\leqq d + 2$; and,

(iv) all compact convex subsets of X.

To show the independence of Axiom 2 more effort is required. First, we observe the result: If Q denotes the set of all rational points in E^2 and f is a similitude of E^2, then $Q \cap f[Q]$ must be either $\emptyset$, a singleton, or Q itself. This is proved by using the analytic form of a plane similitude (the plane being coordinatized by (ξ, η)):

$$\xi' = \alpha\xi - \beta\eta + \lambda ,$$
$$\eta' = \varepsilon\beta\xi + \varepsilon\alpha\eta + \mu ,$$

where $\varepsilon = \pm 1$. Then if (ξ_i, η_i), $i = 1, 2$, are points of $Q \cap f[Q]$ it may be readily shown that α and β are rational and that, therefore, λ and μ are rational (call f *rational* in this case). Hence $f[Q] = Q$, and the result follows.

EXAMPLE 3. With $X = E^2$, define $\mathscr{C}$ as the collection consisting of $\emptyset$, X, Q, all sets of the form $f[Q]$, where $f \in \Omega \equiv$ family of plane similitudes, and all singleton subsets of X.

It may be proved that $\mathscr{C}$ is a convexity structure (to show that $S = \bigcap_{i \in I} f_i[Q]$ is a member of $\mathscr{C}$ if $f_i \in \Omega$ for each $i \in I$, write $f_j^{-1}[S] = \bigcap_{i \in I} g_i[Q]$, where $g_i = f_j^{-1}f_i$, and thus $g_j[Q] = Q$; apply the above observation to show that either $f_j^{-1}[S]$ is a singleton or g_i is rational for all i). Axiom 1 is valid, Axiom 2 is obviously denied, and Axiom 3 holds trivially since if $C \in \mathscr{C}$ has 2 or more points, $\operatorname{cl} C = X$. Because of the pathological nature of the example it is remarkable that $c = 3$, $h = 3$, and $r = 4$—as in classical convexity! (Thus, Axioms 4 and 5 are valid.) Leaving the other two proofs for the reader, consider the following argument for $h = 3$: Let $C_1, \cdots, C_k$ be members of $\mathscr{C}$ such that any 3 intersect (it is obvious that $h \neq 2$ by considering the $\mathscr{C}$-hulls of the pairs of three points consisting of two rationals and one irrational point). Without loss of generality, assume that $C_i \neq X$ and $C_i \neq$ singleton, for each i. Thus $C_i = f_i[Q]$, $f_i \in \Omega$.

Case 1. $C_1 \cap C_i = x$ for some i. Then let $j \neq i$ and consider $C_1 \cap C_i \cap C_j \neq \emptyset$. Thus $x \in C_j$ and $x \in \bigcap_{i=1}^{k} C_i$.

Case 2. $C_1 \cap C_i$ contains at least two points for each i. Then the set $f_1^{-1}[C_1 \cap C_i] = Q \cap f_1^{-1}f_i[Q]$ contains at least two points so that $f_1^{-1}[C_1 \cap C_i] = Q$ or $C_1 \cap C_i = f_1[Q]$, for all i. Therefore, $f_1[Q] \subset \bigcap_{i=1}^{k} C_i$, and $\mathscr{C}$ has Helly number 3.

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