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SUBALGEBRA SYSTEMS OF POWERS OF PARTIAL UNIVERSAL ALGEBRAS

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A set A and an integer $n > 1$ are given. S is any family of subsets of A^n. Necessary and sufficient conditions are found for the existence of a set F of finitary partial operations on A such that S is the set of all subalgebras of $\langle A; F\rangle^n$. As a corollary, a family E of equivalence relations on A is the set of all congruences on $\langle A; F\rangle$ for some F if and only if E is an algebraic closure system on A^2.

For any partial universal algebra, the subalgebras of its nth direct power form an algebraic lattice. The characterization of such lattices for the case $n = 1$ was essentially given by G. Birkhoff and O. Frink [1]. For the case $n = 2$, the characterization was given by the author [4] (see also [3]). The connection between the subalgebra lattices of partial universal algebras and their direct squares was described by the author [5].

In the present paper we are concerned with the subalgebra systems from the following point of view: given a set A and a positive integer n, which systems of subsets of A^n are the subalgebra systems of $\langle A; F\rangle^n$ for some set of partial operations F on A? The problem where F is required to be full is Problem 19 of G. Gratzer [2]. For $n = 1$, such systems are precisely the algebraic closure systems on A[1]. The description of the case $n \geqq 2$ is given here by the Characterization Theorem. We also show that there are partial universal algebras $\langle A; F\rangle$ such that the subalgebra system of $\langle A; F\rangle^2$ is not equal to the subalgebra system of $\langle A; G\rangle^2$ for any set of full operations G. The methods of this paper can be modified to get similar results for infinitary partial algebras, the arities of whose operations are less than a given infinite ordinal.

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1. Let A be a set and n be a positive integer. The set of all functions from $\{1, \cdots, n\}$ into A will be denoted by A^n. If F is a set of finitary partial operations on A, the partial algebra structure obtained on A will be denoted by $\langle A; F\rangle$. By $\langle A; F\rangle^n$ we will mean the partial algebra $\langle A^n; F\rangle$ such that if $f \in F$ is an m-ary partial operation and $a_1, \cdots, a_m \in A^n$ then $a_1 \cdots a_m f$ is defined and equal to $a \in A^n$ if and only if $a_1(j) \cdots a_m(j) f$ is defined and is equal to $a(j)$ for all $1 \leqq j \leqq n$. By a subalgebra of a partial algebra $\langle A; F\rangle$ we will mean

a nonvoid subset of A which is closed under all elements of F. We denote the set of all subalgebras of a partial algebra $\langle A; F\rangle$ by $S(\langle A; F\rangle)$ and we will consider $\phi \in S(\langle A; F\rangle)$ if and only if the intersection of all nonvoid subalgebras of $\langle A; F\rangle$ is empty.

PROPOSITION 1. *$S(\langle A; F\rangle^n)$ is an algebraic closure system on A^n.*

If $n = 1$, this follows from the result of G. Birkhoff and 0. Frink [1]. For any positive n, $S(\langle A; F\rangle^n) = S(\langle A^n; F\rangle)$.

We shall consider only the case $n \geqq 2$.

2. Let S_n be the group of all permutations of $\{1, \cdots, n\}$. Denote by $P(A^n)$ the set of all subsets of A^n. If $s \in S_n$ and $B \in P(A^n)$, we define

$$Bs = \{a\colon\ a \in A^n,\ b, b \in B, a(i) = b(is^{-1}), 1 \leqq i \leqq n\}. \tag{1}$$

For $n = 2$, $B \subseteqq A^2$, $B(12)$ is the inverse binary relation of B.

PROPOSITION 2. *The mapping which associates to every $s \in S_n$ the operator on $P(A^n)$ defined by (1) is a group homomorphism of S_n into the group of all automorphisms of the lattice $\langle S(\langle A; F\rangle)^n; \subseteqq\rangle$.*

3. Let α be a nonvoid subset of $\{1, \cdots, n\}$, $i = \min \alpha$ and $B \in P(A^n)$. Define

$$\begin{aligned} B\alpha = \{a\colon\ a \in A^n,\ b \in B, a(j) = b(j) \ \textit{if}\ j \notin \alpha, \\ a(j) = b(i) \ \text{if}\ j \in \alpha, 1 \leqq j \leqq n\}. \end{aligned} \tag{2}$$

It is easy to verify that if $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$ then

$$B\{i_1, \cdots, i_k\} = (\cdots((B\{i_1, i_2\})\ \{i_2, i_3\})\cdots)\ \{i_{k-1}, i_k\}. \tag{3}$$

If $C \in P(A^n)$, we denote by $F(C)$ the subalgebra of $\langle A; F\rangle^n$ generated by C.

PROPOSITION 3. *If $C \in P(A^n)$, α — a nonvoid subset of $\{1, \cdots, n\}$ and $s \in S_n$, then*

$$F(C)\alpha \subseteqq F(C\alpha) \tag{4}$$

$$F(C)s = F(Cs)\ . \tag{5}$$

4. We denote by Δ_k the diagonal of A^k,$B \times A^\circ$ will be identified with B.

The Characterization Theorem. Let $S \subseteqq P(A^n)$. $S = S(< A; F\rangle^n)$ for some set of finitary partial operations F if and only if

(a) *S is an algebraic closure system on A^n*
(b) *if $B \in S$, $1 \leqq i < j \leqq n$, then $B(ij) \in S$.*
(c) $\Delta_2 \times A^{n-2} \in S$
(d) $[C]\ \{1, 2\} \subseteqq [C\{1, 2\}]$ *for all nonvoid finite $C \in P(A^n)$*
(e) *if $\phi \in S$, then $\phi = \cap \{B: \phi \neq B \in S\}$.*
(*$[C]$ denotes the intersection of all elements of S containing C*).

It can be shown that conditions (a), (b), (c), (d) and (e) are independent.

It is clear that $\Delta_2 \times A^{n-2}$ is a subalgebra of $\langle A; F\rangle^n$ for all F. That conditions (a), (b) and (d) are necessary follows from Propositions 1, 2 and 3.

Proof of Sufficiency. For every positive integer m and every ordered $m+1$ – tuple $(a_1, \cdots, a_m, a)$ of elements of A^n such that $a \in [\{a_1, \cdots, a_m\}]$ we associate an m-ary partial operation f on A such that

Df = domain of definition of $f = \{(a_1(i), \cdots, a_m(i)): 1 \leqq i \leqq n\}$ and

$$a_1(i) \cdots a_m(i) f = a(i), 1 \leqq i \leqq n.$$

Let F be the set of all such finitary partial operations.

The following lemmas constitute the proof of sufficiency:

LEMMA 1. *If $C \in P(A^n)$, $s \in S_n$, then $[C]s = [Cs]$.*

By (a), S is a closure system hence $[C] \in S$. From (b) $[C](ij) \in S$ for all $1 \leqq i < j \leqq n$. Hence, by Proposition 2, $[C]s \in S$ $(P(A^n) = S(\langle A; \phi\rangle^n))$. *But*

$$Cs \subseteqq [C]s \in S .$$

Hence

$$[Cs] \subseteqq [C]s .$$

Also

$$C = (Cs)s^{-1} .$$

Hence

$$[C] = [(Cs)s^{-1}] \subseteqq [Cs]s^{-1} .$$

And so

$$[C]s \subseteqq [Cs] .$$

LEMMA 2. *If α is a nonvoid subset of $\{1, \cdots, n\}$ and $C \in P(A^n)$, C is finite and nonvoid, then*

$$[C]\alpha \subseteqq [C\alpha] .$$

First we show Lemma 2 for the case $\alpha = \{i, j\}, 1 \leqq i < j \leqq n$.

$$\begin{aligned} [C]\ \{i, j\} &= (([C](1i)(2j))\ \{1, 2\})(1i)(2j) \\ &= ([C(1i)(2j)]\ \{1, 2\})(1i)(2j) && \text{(by Lemma 1)} \\ &\subseteqq [(C(1i)(2j)\ \{1, 2\})(1i)(2j) && \text{by } (d) \\ &= [(C(1i)(2j)\ \{1, 2\})(1i)(2j)] \\ &= [C\{i, j\}]\ . \end{aligned}$$

If $1 \leqq i_1 < \cdots < i_k \leqq n$, then

$$\begin{aligned} [C]\ \{i_1, \cdots, i_k\} &= (\cdots(([C]\ \{i_1, i_2\})\ \{i_2, i_3\})\cdots)\ \{i_{k-1}, i_k\} && \text{(by (3))} \\ &\subseteqq (\cdots([C\{i_1, i_2\}]\{i_2, i_3\})\cdots)\ \{i_{k-1}, i_k\} \\ &\subseteqq \cdots \\ &\subseteqq [C\{i_1, \cdots, i_k\}]\ . \end{aligned}$$

LEMMA 3. *The definition of F is correct, i.e. every $f \in F$ is one valued.*

Lemma 3 will be established once we show that whenever $a_1, \cdots, a_m \in A^n, f \in F$ are such that $a_1(i)\cdots a_m(i)f$ is defined for every $1 \leqq i \leqq n$ and if for some $1 \leqq p < q \leqq n$ $a_1(p) = a_1(q), \cdots, a_m(p) = a_m(q)$; then

$$a_1(p)\cdots a_m(p)f = a_1(q)\cdots a_m(q)f\ .$$

By the definition of F, there are $c_1, \cdots, c_m, c \in A^n$ such that $c \in [\{c_1, \cdots, c_m\}]$,

$$Df = \{(c_1(i), \cdots, c_m(i))\colon 1 \leqq i \leqq n\}$$

and

$$c_1(i)\cdots c_m(i)f = c(i);\ 1 \leqq i \leqq n\ .$$

Hence

$$\begin{aligned} \{(a_1(i), \cdots, a_m(i))\colon 1 \leqq i \leqq n\} &\subseteqq Df \\ &= \{(c_1(i), \cdots, c_m(i))\colon 1 \leqq i \leqq n\}\ . \end{aligned}$$

So there are $s \in S_n$ and α nonvoid subset of $\{1, \cdots, n\}$ such that

$$a_t = c_t s\alpha, 1 \leqq t \leqq m\ .$$

Since every a_t satisfies $a_t(p) = a_t(q)$. We have $a_t \in (\varDelta_2 \times A^{n-2})(1p)(2q) \in S$ (by (c) and (b)) for all $1 \leqq t \leqq m$. Then

$$\{c_1, \cdots, c_m\}s\alpha = \{a_1, \cdots, a_m\} \subseteqq (\varDelta_2 \times A^{n-2})(1p)(2q)\ .$$

But

$$[\{c_1, \cdots, C_m\}]s\alpha = [\{c_1, \cdots, c_m\}s]\alpha$$

$$\begin{aligned} &\subseteqq [\{c_1, \cdots, c_m\}s\alpha] \\ &= [\{a_1, \cdots, a_m\}] \\ &\subseteqq (\varDelta_2 \times A^{n-2})(1p)(2q) . \end{aligned}$$

Define $a \in A^n$ by

$$a(j) = a_1(j)\cdots a_m(j)f \qquad 1 \leqq j \leqq n .$$

Then

$$a = cs\alpha \in [\{c_1, \cdots, c_m\}]s\alpha \subseteqq (\varDelta_2 \times A^{n-2})(1p)(2q) .$$

Hence

$$a_1(p)\cdots a_m(p)f = a(p) = a(q) = a_1(q)\cdots a_m(q)f .$$

LEMMA 4. *If $\phi \neq B \in S(\langle A; F\rangle^n)$ then $B \in S$.*

Since S is an algebraic closure system it will be sufficient to show that if C is a finite nonvoid subset of B, the $[C] \subseteqq B$.

Let $b_1, \cdots, b_m \in B$ and $b \in [b_1, \cdots, b_m]$. By the definition of F, there is $f \in F$ such that $b_1(i)\cdots b_m(i)f$ is defined and is equal to $b(i)$ for all $1 \leqq i \leqq n$. B is a subalgebra of $\langle A; F\rangle^n$, hence $b \in B$.

LEMMA 5. *If $\phi \neq B \in S$ then $B \in S(\langle A; F\rangle^n)$.*

Let $f \in F$; $a_1, \cdots, a_m \in B$ and $a_1(i)\cdots a_m(i)f = a(i), 1 \leqq i \leqq n$. We must show that $a \in B$.

By the definition of F there are $c_1, \cdots, c_m, c \in A^n$ such that $c \in [\{c_1, \cdots, c_m\}]$

$$Df = \{(c_1(i), \cdots, c_m(i))\colon 1 \leqq i \leqq n\}$$

and

$$c_1(i)\cdots c_m(i)f = c(i), 1 \leqq i \leqq n .$$

So

$$\begin{aligned} &\{(a_1(i), \cdots, a_m(i))\colon 1 \leqq i \leqq n\} \subseteqq Df \\ &= \{(c_1(i), \cdots, c_m(i))\colon 1 \leqq i \leqq n\} . \end{aligned}$$

As in Lemma 3

$$a_t = c_t s\alpha, 1 \leqq t \leqq m;\ a = csd ,$$

for some $s \in S_n$ and $\phi \neq a \subseteqq \{1, \cdots, n\}$.
But

$$c \in [\{c_1, \cdots, c_m\}] .$$

Hence

$$\begin{aligned} a = cs\alpha \in [\{c_1, \cdots, c_m\}]s\alpha &\subseteqq [\{c_1, \cdots, c_m\}s\alpha] \\ &= [\{a_1, \cdots, a_m\}] \subseteqq B . \end{aligned}$$

THEOREM 5. *Let $C \subseteqq P(A^2)$. C is the set of all congruence* *relations on $\langle A; F\rangle$ for some set of finitary partial operations F if and only if C is an algebraic closure system on A^2 and every element of C is an equivalence relation on A.*

That the set of all congruence relations on $\langle A; F\rangle$ is an algebraic closure system on A^2 is well known.

If $C \subseteqq P(A^2)$ is a set of equivalence relations which is also an algebraic closure system on A^2 then C satisfies all the conditions (a), (b), (c), (d) and (e) of the Characterization Theorem. Hence $C = S(\langle A; F\rangle^2)$ for some set of finitary partial operations F. Since every element of C is an equivalence relation on A and a subalgebra of $\langle A; F\rangle^2$, it is a congruence relation on $\langle A; F\rangle$. Since a congruence relation on $\langle A; F\rangle$ is an equivalence relation on A which is also a subalgebra of $\langle A; F\rangle^2$, the Theorem is proved.

6. The following proposition shows that our Characterization Theorem does not solve the corresponding problem for full algebras.

PROPOSITION 4. *There are partial algebras $\langle A; F\rangle$ such that $S(\langle A; F\rangle^2) \neq S(\langle A; G\rangle^2)$ for any set of full finitary operations G.*

Let $A = \{1, 2, 3\}$, $F = \{f_1, f_2, f_3, g\}$; f_1, f_2, f_3 are full unary operations, g is a partial binary operation.

f_i is the constant function taking the value i, $i = 1, 2, 3$.

$$Dg = \{(1, 2), (2, 1)\}$$
$$12g = 3,\ (21)g = 2,$$
$$B = \Delta_2 \cup \{(1, 2)\},\ C = \Delta_2 \cup \{(2, 1)\}$$
$$BoC = B \cup C$$
$$B, C \in S(\langle A; F\rangle^2),\ \text{but}$$
$$BoC \notin S(\langle A; F\rangle^2)$$

since any subalgebra of $\langle A; F\rangle^2$ containing (1, 2) and (2, 1) contains also (3, 2) and (2, 3).

REFERENCES

1. G. Birkhoff and 0. Frink, *Representations of Lattices by sets*, Trans. Amer. Math. Soc., **64**(1948), 299-316.

2. Grätzer, *Universal Algebra*, Vand Nostrand, Princeton, N.J., (1967).

3. G. Grätzer and W. A. Lampe, *On the Subalgebra lattice of universal algebras*, J. Algebra, **7** (1967), 263-270.

4. A. A. Iskander, *Correspondence lattices of univesal algebras*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat., **29**(1965), 1357-1372.

5. ———, *Partial universal algebras with preassigned lattices of subalgebras and correspondences* (Russian) Mat. Sb., (N.S.) **70** (112) (1966), 438-456.

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