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A NOTE ON TWO GENERALIZATIONS OF QF-3

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If M is an R-module, then the dual of M is defined to be $\mathrm{Hom}_R(M, R)$. Artinian QF-3 rings R have been characterized by the following two properties:

(1) The class of R-modules with zero duals is closed under taking submodules.

(2) The class of torsionless R-modules is closed under extension.

These properties are independent and, in the present paper, we study the two classes of rings R which satisfy each of these conditions separately.

Let R be a ring with identity. R is said to be (left) QF–3 provided there is an idempotent e in R such that Re is faithful and injective as a (left) R-module. The notion of QF–3 rings is derived from the definition of QF–3 algebras introduced by Thrall in [4].

If M is a left R-module, let $M^* = \mathrm{Hom}(M, R)$ denote the "dual" of M, with the usual right module structure. For left Artinian rings R, Wu, Mochizuki and Jans [5] have given the following two properties characterizing those which are QF–3.

(1) If $M_1 \subseteqq M_2$ are R-modules, then $M_2^* = (0)$ implies $M_1^* = (0)$.

(2) The class of torsionless R-modules is closed under extension.

That is, if A and C are torsionless R-modules, and $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then B is torsionless.

In this note, rings satisfying (1) or (2) separately are studied. Those satisfying (1) are called SZD and those satisfying (2), TCE. For (left) R-modules M, the following notation is used,

$Z(M) = \{m \in M \mid Em = 0$ for some essential left ideal $E \subseteqq R\}$ (the singular submodule of M)

$S(M) =$ the sum of all simple submodules of M (the socle of M)

$E(M) =$ injective hull of M

SZD and TCE Rings

PROPOSITION 1. *A ring R is SZD if and only if the following are equivalent for every R-module M.*

(1) $\mathrm{Hom}(M, R) = (0)$ (2) $\mathrm{Hom}(M, E(R)) = (0)$

Proof. Assume R is SZD. Condition (2) implies (1) trivially. To show (1) implies (2), assume $M^* = (0)$ and let $f \neq 0$ in $\mathrm{Hom}(M, E(R))$. Set $L = f(M) \cap R$ and $M_0 = f^{-1}(L)$. Then $M_0 \neq (0)$ and $f|_{M_0}: M_0 \to R$ is nonzero, so that $M_0^* \neq (0)$. Since R is SZD, this implies $M^* \neq (0)$, a contradiction.

Conversely, if (1) $\Leftrightarrow$ (2), let $M^* = 0$. If M_0 is a submodule of M, we have $\operatorname{Hom}(M, R) = (0) \Rightarrow \operatorname{Hom}(M, E(R)) = (0) \Rightarrow \operatorname{Hom}(M_0, E(R)) = (0) \Rightarrow \operatorname{Hom}(M_0, R) = (0)$.

PROPOSITION 2. *If R is* SZD *and $Z(R) = (0)$, then $E(R)$ is torsionless.*

Proof. Let $K = \bigcap_{f \in \operatorname{Hom}(E(R), R)} \operatorname{Ker} f$, and assume $K \neq (0)$. Then $\operatorname{Hom}(E(K), R) \neq (0)$ since R is SZD. Choose $f \neq 0$ in $\operatorname{Hom}(E(K), R)$ and pick $x \in E(K)$ such that $f(x) \neq 0$. Set $A = \{r \in R \mid rf(x) = 0\}$. Because $Z(R) = (0)$, A is not essential in R, and there is a left ideal $L \neq (0)$ such that $L \cap A = (0)$. Then $Lx \cap K \neq (0)$, so there is a $r \in L$ such that $rx \in K$ and $f(rx) \neq 0$. But $E(R) = E(K) \oplus Y$ for some $Y \subseteqq E(R)$, and f can be extended to $\bar{f}: E(R) \to R$, contradicting the definition of K.

COROLLARY. *If R is* SZD *and $Z(R) = (0)$, then an R-module M is torsionless if and only if $E(M)$ is torsionless.*

Proof. If $E(M)$ is torsionless, M is a torsionless submodule. If M is torsionless, M can be embedded in a product, πR, of copies of R. Then $E(M)$ can be embedded in a product, $\pi E(R)$, of copies of $E(R)$. Since $E(R)$ is torsionless by Prop. 2, so is $\pi E(R)$, and hence $E(M)$ is torsionless.

COROLLARY. *If R is* SZD *and $Z(R) = 0$, then R is* TCE.

Proof. Kato [2] has observed that the proof in [5] can be modified slightly to show that in any ring, SZD and TCE are equivalent to $E(R)$ being torsionless. Hence, by Prop. 2, R must be TCE.

THEOREM 3. *If R is right perfect and $Z(R) = (0)$, then* SZD *implies QF-3.*

Proof. Tachikawa [3] has shown that in a right perfect ring, $E(R)$ torsionless implies R is QF-3.

We continue with some results on TCE rings. For an R-module M we let j_M denote the natural map from M to its double dual M^{**}.

THEOREM 4. *If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence of R-modules with A and C torsionless, then B is torsionless if and only* *if* $\operatorname{Im} j_A \cap \operatorname{Ker} \alpha^{**} = (0)$, *where* α^{**} *is the induced map from* A^{**} *to* B^{**}.

Proof. Apply the exact sequence in Ext to $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, to obtain an exact sequence $0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \xrightarrow{\delta} X \to 0$, where $X \subseteqq \operatorname{Ext}^1_R(C, R)$ is the image of A^* under the connecting map δ (see [1]). Take the dual of the latter sequence to obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^* & \longrightarrow & A^{**} & \xrightarrow{\alpha^{**}} & B^{**} & & \\ & & & & \uparrow{\scriptstyle j_A} & & \uparrow{\scriptstyle j_B} & & \\ & & 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & C \longrightarrow 0 \end{array}$$

a row exact, commuting diagram. Now B torsionless implies $j_B\alpha$ is monic, so that $\alpha^{**}j_A$ is monic, and $\operatorname{Ker} \alpha^{**} \cap \operatorname{Im} j_A = (0)$. Conversely, if $\operatorname{Ker} \alpha^{**} \cap \operatorname{Im} j_A = (0)$, then $\operatorname{Ker} j_B \cap \operatorname{Im} \alpha = (0)$. Thus if $0 \neq b \in \operatorname{Ker} j_B$, $\beta(b) \neq 0$. Since C is torsionless, there is a map $f: C \to R$ such that $f(\beta(b)) \neq 0$. But then $f\beta: B \to R$ satisfies $(f\beta)(b) = 0$, contradicting $b \in \operatorname{Ker} j_B$ (see [1]). Therefore $\operatorname{Ker} j_B = (0)$ and B is torsionless.

Theorem 4 says R is TCE if and only if $\operatorname{Im} j_A \cap \operatorname{Ker} \alpha^{**} = (0)$ for every short exact sequence $0 \to A \xrightarrow{\alpha} B \to C \to 0$ with A and C torsionless.

We now define a special type of torsionless R-module for use in further investigation of TCE rings.

DEFINITION. An R-module $M \neq (0)$ is completely torsionless (c.t.) provided M is torsionless and has no nontrivial torsionless factors.

It is immediate that a c.t. module M must be isomorphic to a left ideal of R, for there must be a nonzero map $f: M \to R$ since M is torsionless, and $\operatorname{Ker} f = (0)$ since M has no torsionless factors.

LEMMA 5. *If R is left Noetherian, every left ideal has a completely torsionless factor.*

Proof. Let L be a left ideal in R. If L is not c.t., L has a torsionless factor L/L_1. If L/L_1 is not c.t., there is left ideal $L_2 \supseteqq L_1$ such that L/L_2 is torsionless. Continuing in this fashion we obtain an ascending chain $\{L_i\}$ of left ideals which must terminate. That is, L/L_n is c.t. for some n.

COROLLARY. *If R is left Noetherian and M is an R-module with* $\operatorname{Hom}(M, R) \neq (0)$, *then M has a completely torsionless factor.*

Proof. Pick $f \neq 0$ in $\mathrm{Hom}(M, R)$. Then $f(M)$ has a c.t. factor, so M does also.

THEOREM 6. *If R is left Noetherian, R is* TCE *if and only if every short exact sequence $0 \to A \to B \to C \to 0$ with C torsionless and A completely torsionless, must have B torsionless as well.*

Proof. The "only if" part follows from the definition of TCE. To show the "if" part, let $0 \to A \to B \to C \to 0$ be exact with A and C torsionless. Set $\bar{M} = \{M \mid M \text{ is a submodule of } A,\ B/M \text{ torsionless}\}$. Then $A \in \bar{M}$, and if $M_1 \supseteq M_2 \supseteq \cdots$ is a descending chain in $\bar{M}$, define a map ϕ: $B/\bigcap_{i=1}^{\infty} M_i \to \prod_{i=1}^{\infty} B/M_i$ by $\phi(b + \bigcap_{i=1}^{\infty} M_i) = \prod_{i=1}^{\infty} (b + M_i)$. It is easy to check that ϕ is an R-monomorphism. Thus, $B/\bigcap_{i=1}^{\infty} M_i$ is isomorphic to a submodule of the torsionless module $\prod_{i=1}^{\infty} B/M_i$. This implies $B/\bigcap_{i=1}^{\infty} M_i$ is torsionless, hence $\bigcap_{i=1}^{\infty} M_i \in \bar{M}$. Now apply Zorn's Lemma to $\bar{M}$ to obtain a minimal element M_0. If M_0 is c.t., then $0 \to M_0 \to B \to B/M_0 \to 0$ gives B torsionless by hypothesis. If M_0 is not c.t., there is a completely torsionless factor M_0/N by Corollary to Lemma 5. But the exact sequence $0 \to M_0/N \to B/N \to B/M_0 \to 0$ implies that B/N is torsionless, contradicting the minimality of M_0 in $\bar{M}$. Thus M_0 is in fact c.t., and B is torsionless.

We next consider short exact sequences where the factor module is c.t. The theorem in this case is only for finitely generated modules over left Artinian rings.

THEOREM 7. *Let R be left Artinian. The class of finitely generated torsionless modules is closed under extension if and only if every exact sequence of finitely generated modules, $0 \to A \to B \to C \to 0$, with C completely torsionless and A torsionless, has B torsionless as well.*

Proof. Again the "only if" part is immediate. For the "if" part, let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be exact with A and C finitely generated and torsionless. If C is not c.t., there is a c.t. factor C/D_1. Let $A_1 = \beta^{-1}(D_1) \supseteq A$. This gives $0 \to A_1 \to B \to C/D_1 \to 0$ exact. If A_1 is torsionless, then so is B by hypothesis. If not, consider $0 \to A \to A_1 \to D_1 \to 0$. Let D_1/D_2 be a c.t. factor of D_1, and $A_2 = \beta^{-1}(D_2) \subseteq A_1$. This yields $0 \to A_2 \to A_1 \to D_1/D_2 \to 0$ exact. If A_2 is torsionless, then A_1 is also, contradiction. The process may be continued inductively, obtaining at each stage D_{n-1}/D_n completely torsionless and $A_n = \beta^{-1}(D_n)$. The sequence $C \cong B/A \supseteq A_1/A \supseteq A_2/A \supseteq \cdots$ must terminate since C is finitely generated and R is left Artinian. By construction, the sequence stops at A_n/A if and only if A_n is torsionless. But A_n torsionless implies A_{n-1} torsionless, also by construction. We conclude that A_1, hence B, is torsionless.

Note that the above proof does not require that A be finitely generated, and the theorem can be generalized slightly. Consideration of short exact sequences with c.t. modules at both ends failed to yield any significant results.

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