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A MAP-THEORETIC APPROACH TO DAVENPORT-SCHINZEL SEQUENCES

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An (n, d) Davenport-Schinzel Sequence (more briefly, a DS sequence) is a sequence of symbols selected from $1, 2, \cdots, n$, with the properties that (1) no two adjacent symbols are identical, (2) no subsequence of the form abab ... has length greater than d, (3) no symbol can be added to the end of the sequence, without violating (1) or (2). It is shown that the set of $(n, 3)$ DS sequences is in one-to-one correspondence with the set of rooted planar maps on n vertices in which every edge of the map is incident with the root face. The number of such sequences and the number of such sequences of longest possible length $2n - 1$ is explicitly determined.

As an illustration of DS sequences, take $n = 4$, $d = 3$. Then it is simple to enumerate all DS sequences in normal form (the symbols occur in increasing order). The results follow, and we see that there are 11 (4, 3) DS sequences.

1213141
121341
1213431
123141
1232141
123241
1232421
123421
123431
1234321
123431.

It is obvious from the definition that a normal $(n, 3)$ DS sequence must begin and end with 1.

The concept of a DS sequence was introduced in [1], and various results were obtained. For subsequent developments, one may consult [2], [3], and [4].

Among all DS sequences for fixed n and d, some will have greatest lengths. We define the number $N_d(n)$ to be this greatest length. For example, in the preceding example, $N_3(4) = 7$, and there are 5 sequences of this greatest length.

The results of [1], [2], [3], [4], basically concern the value of $N_d(n)$. In this paper, we consider $(n, 3)$ sequences (it was shown in [1] that $N_3(n) = 2n - 1$). We set up a correspondence with rooted planar maps, obtain an enumeration of $(n, 3)$ DS sequences, and show that the number of DS sequences of greatest length is given by the familiar Catalan sequence 1, 1, 2, 5, 14, $\cdots$.

2. Face maps. By a *rooted* planar map we mean a planar mpa which is either the vertex map or a map in which an edge is distinguished as the root edge, and positive and negative ends and left and right sides are specified for this root edge. The vertex at the negative end is called the root vertex, and the face on the left is called the root face. For convenience, we shall henceforth assume that the map has no loops or multiple joins. *A face map* is rooted planar map in which every edge is incident with the root face. (Thus, apart from the rooting, the map can be considered to be a typical face of any planar map.)

Given any face map, we can obtain a sequence of integers, the *face sequence*, from it as follows. If the map is the vertex map, the sequence is 1. Otherwise the integer 1 is assigned to the root vertex. Then, in the root face, starting with the root vertex and proceeding along the root edge, the next vertex encountered is labelled 2. Every time an unlabelled vertex is encountered, it is assigned the least positive integer which has not been used as a label; this is done until one returns to the root vertex. The sequence is then constructed by listing the labels as they are encountered as one traces out the root face, beginning $1, 2, \cdots$, and ending with the symbol 1, when the root vertex is encountered for the last time before retracing the root edge. With the exception of vertex map, such a sequence will have length $n + 2$, where n is the number of edges in the map, isthmuses being counted twice.

We now point out a $1 - 1$ correspondence between face sequences and normal Davenport-Schinzel sequences of type $(n, 3)$. Indeed, consider any face map. As it is traversed in the above fashion, any unordered pair of distinct vertices is linked by a unique sequence of edges, which we shall call the arc ab, which is traversed precisely once. The arc ab is, of course, identical with the arc ba. Now if the face sequence of a face map were to contain a subsequence $a\ b\ a\ b$, for $a \neq b$, then the arc ab would be traced twice, and this would be a contradiction. Also, since there are no loops, no 2 consecutive symbols are identical. Hence properties (1) and (2) of DS properties are satisfied. Also, every vertex occcurs in the sequence; thus the addition of any symbol a other than 1 introduces a subsequence 1 a

1 a, and this violates (2); on the other hand, the addition of 1 violates (1). Hence DS property (3) is also satisfied, and we sum up our results in

THEOREM 1. *Every face sequence is a DS sequence.*

We now demonstrate the converse result, namely.

THEOREM 2. *Every normal DS sequence is the face sequence of a face map.*

Proof. We use induction on the number of symbols in a normal DS sequence. The result is trivially true for the set A_1 of normal DS sequences on the symbol 1. Indeed, A_1 has only one member, the sequence 1, which corresponds to the vertex map. Similarly for A_2, the set of normal DS sequences on 12, which contains only 121, which corresponds to the rooted edge.

Nor let us assume that the result is valid for A_i, the set of normal DS sequences on $1, 2, \cdots$, for $i \leqq n-1$. We establish the result for A_n. Clearly the sequence $1, 2, 3, \cdots, n, 1$, is a member of A_n, and corresponds to the rooted $n-$ gon. Let X be any other member of A_n. It is evident that X contains either at least three l's or some repeated member of $2, 3, \cdots, n$. If X contains three or more l's, then write $X = PQ$, where $P = 1, 2, \cdots, 1$, $Q = w, \cdots, 1$, and P contains only two l's. Let $R = IQ$, and let S be the normalized version of R. For example, if X is $1\,2\,3\,1\,4\,1\,5\,1$, then P is $1\,2\,3\,1$, $Q = 4\,1\,5\,1$, $R = 1\,4\,1\,5\,1$, and S is $1\,2\,1\,3\,1$. Evidently R and S are DS sequences on fewer than n symbols, and as such correspond to face maps M and N respectively. Let x and y be the root vertices of M and N respectively, and (xa) and (yb) be the root edges of M and N respectively. Choose vertices c and d such that cxa and dyb are the angles of M and N respectively which lie to the left of the root edges. (Occasionally this may also be the right.) We then embed N homeomorphically in the root face of M, identifying x and y, and carry out the embedding so that N lies in the angle cxa, and so that $c(xy)b$ is an angle of the resultant map, P, where (xy) is the vertex obtained by identifying x and y. The root edge, together with its positive and negative ends and left and right sides, is taken as the rooting for P. It is clear that P is a face map, and that its face sequence is X. If, on the other hand, X contains only two l's, and some other symbol occurs twice, let a be the smallest symbol greater than 1 which is repeated. Write X as PQR, where Q is of the form $ab \cdots a$, $b \neq a$, and Q contains only two $a's$, whereas P contains none. Let $T = PSR$, and let U and V be the normalized version of Q and

T respectively. Let M and N be the face maps of T and Q respectively. Then by embedding N in the root face of M at the angle of M corresponding to the first occurrence of a in T, in a fashion analogous to that described previously, one obtains a map P whose face sequence is X. It may be verified that distinct DS sequences give rise to distinct maps in the above construction. Thus, if Y_i is the set of face sequences corresponding to maps with i vertices, $|A_i| \leqq |Y_i|$; however, since every face sequence is DS, then $|Y_i| \leqq |A_i|$, and the two sets are in $1-1$ correspondence. Indeed, we have shown that a sequence is a face sequence if and only if it is DS.

COROLLARY. *The greatest length for a DS sequence on n symbols is $2n-1$; such sequences correspond to rooted planar trees, in which every edge is an isthmus.*

3. **The number of DS sequences.** It is well known that the number of topologically distinct rooted plane trees on n vertices is given by the Catalan number $(2n-2)!/(n-1)!\,n!$; thus this is the number of DS sequences of greatest length on n symbols. We also determine the number of normal DS sequences on n symbols by enumerating face maps with n vertices.

Let f_n represent the number of such face maps, and define a generating function $F(x) = \sum_{n=1}^{\infty} f_n x_n$. Let F_k be the set of face maps whose root edge is contained in a k-gon. (We let $k=2$ correspond to the case in which the root edge is an isthmus.) Every member of F_k is completely determined by the ordered set of maps which occur in the angles of the k-gon as one proceeds around it in the direction induced by the root. Thus the generating function for F_k is $(F(x))^k$. Hence $F(x)$ satisfies the equation

$$F(x) = x + \sum_{k=2}^{\infty} (F(x))^k \,, \tag{1}$$

where the term x corresponds to the vertex map.

We than see that

$$F(x) = x + (F(x))^2/(1-F(x)) \,. \tag{2}$$

Hence,

$$2(F(x))^2 - (1+x)F(x) + x = 0 \,, \tag{3}$$

$$F(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4} \,, \tag{4}$$

where the radical denotes the series with constant term 1. The coefficients f_n may be determined by direct expansion of (4), or by applying Lagrange's theorem to (2). In each case, the resulting expression is a finite summation with alternating signs, which is undesirable for calculation. Thus we use another approach, based on the fact that $y = F(x) - 1 - x$ satisfies the differential equation

$$(1 - 6x + x^2)y' - y(x - 3) = 0 . \tag{5}$$

If we use the standard method for obtaining a series solution for (5), where a_n is the coefficient of x^n, we find that a_n satisfies the difference equation

$$(n + 1)a_{n+1} - (6n - 3)a_n + (n - 2)a_{n-1} = 0 . \tag{6}$$

By using initial conditions $a_0 = -1/4$, $a_1 = 3/4$, then, for $n \geqq 2$, $a_n = f_n$, and thus f_n may be easily calculated for small values of n. We find that $f_1 = 1$, $f_2 = 1$, $f_3 = 3$, $f_4 = 11$, $f_5 = 45$, and $f_6 = 197$. Moreover, using the Laplace method for solving linear difference equations in terms of integrals, one can show that for $n \geqq 2$,

$$f_n = \frac{1}{4\pi} \int_{-2\sqrt{2}}^{2\sqrt{2}} (t + 3)^{n-2} \sqrt{8 - t^2}\, dt . \tag{7}$$

This gives an explicit solution for f_n, $n \geqq 2$. However, using the fact that $\sqrt{8 - t^2}$ is an even function, we find that

$$f_n = \frac{1}{2\pi} \sum_{k=0}^{n-2} \binom{n-2}{2k} 3^{n-2k-2} \int_0^{2\sqrt{2}} t^{2k} \sqrt{8 - t^2}\, dt . \tag{8}$$

But

$$\int_0^{2\sqrt{2}} t^{2k} \sqrt{8 - t^2}\, dt = 8^{k+1} \int_0^{\pi/2} (\sin^{2k} \theta - \sin^{2k+2} \theta) d\theta , \tag{9}$$

by which, in virtue of the fact that

$$\int_0^{\pi/2} \sin^n x\, dx = \sqrt{\pi}\,\Gamma[(n + 1)/2]/2\Gamma(n/2 + 1) ,$$

where $\Gamma(x)$ is the Gamma function, we obtain a formula for f_n, $n \geqq 2$, as a sum of positive terms, namely,

$$f_n = \sum_{k=0}^{\infty} 3^{n-3-2k}\, 2^k \binom{n-2}{2k} \frac{2k!}{k!(k + 1)!} , \tag{10}$$

which is convenient for nonrecursive calculation for small values of n.

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