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EVERY GENERALIZED PETERSEN GRAPH HAS A TAIT COLORING

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Watkins has defined a family of graphs which he calls generalized Petersen graphs. He conjectures that all but the original Petersen graph have a Tait coloring, and proves the conjecture for a large number of these graphs. In this paper it is shown that the conjecture is indeed true.

DEFINITIONS. Let n and k be positive integers, $k \leqq n-1$, $n \neq 2k$. The generalized Petersen graph $G(n, k)$ has $2n$ vertices, denoted by $\{0, 1, 2, \cdots, n-1; 0', 1', 2', \cdots, \cdots, (n-1)'\}$ and all edges of the form $(i, i+1)$, (i, i'), $(i', (i+k)')$ for $0 \leqq i \leqq n-1$, where all numbers are read modulo n. $G(5, 2)$ is the Petersen graph. See Watkins [2].

The sets of edges $\{(i, i+1)\}$ and $\{(i', (i+k)')\}$ are called the outer and inner rims respectively and the edges (i, i') are called the spokes.

A Tait coloring of a trivalent graph is an edge-coloring in three colors such that each color is incident to each vertex. A 2-factor of a graph is a bivalent spanning subgraph. A 2-factor consists of disjoint circuits. A Tait cycle of a trivalent graph is a 2-factor all of whose circuits have even length. A Tait cycle induces a Tait coloring and conversely.

The method that Watkins used in proving that many generalized Petersen graphs have a Tait coloring was to prove that certain color patterns on the spokes induce a Tait coloring. Our method for the remaining cases consists of the construction of 2-factors and of proof that these 2-factors are Tait cycles under appropriate conditions.

We restrict ourselves to the generalized Petersen graphs $G(n, k)$ with the properties:

$$n \text{ odd}, \ n \geqq 7, \ (n, k) = 1, \text{ and } 2 < k < \frac{n-1}{2} .$$

All other cases (and some special instances of the above) were dealt with by Watkins.

We construct three types of 2-factors. The first type is a Tait cycle when k is odd. The second type is a Tait cycle when k is even and $n \equiv 3 (\text{mod } 4)$ and also when k is even and $n \equiv 1 (\text{mod } 4)$ with k^{-1} even. (As $(n, k) = 1$, we define k^{-1} as the unique positive integer $< n$, for which $kk^{-1} \equiv 1 (\text{mod } n)$.) The third type takes care of the remaining graphs.

The principal tool in the proofs is the automorphism φ (henceforth fixed) of $G(n, k)$ defined by $\varphi(i) = n - i$; $\varphi(i') = (n - i)$. In each case φ induces an automorphism (also called φ) of the constructed 2-factor. To facilitate notation we write $n = 2m + 1$.

CONSTRUCTION 1. The subgraph H of $G(n, k)$ has the following edges:

(a) On the outer rim: $(m + k, m + k + 1)$, $(m + k + 1, m + k + 2), \cdots, (n - 1, 0), (0, 1), (1, 2), \cdots, (m - k, m - k + 1)$,
$(m - k + 2, m - k + 3), (m - k + 4, m - k + 5), \cdots$,
$(m + k - 2, m + k - 1)$.

The last line may be written as $(m - k + 2j, m - k + 2j + 1)$, $1 \leqq j \leqq k - 1$.

(b) Spokes: $(m + k, (m + k)')$, $(m - k + 1, (m - k + 1)')$, $(m - k + 2, (m - k + 2)')$, $\cdots$ $(m + k - 1, (m + k - 1)')$.

(c) On the inner rim: $(i', (i + k)')$, $m + 1 \leqq i \leqq n - 1$
$(i', (i - k)')$, $k \leqq i \leqq m$.

EXAMPLE. $G(11, 3)$

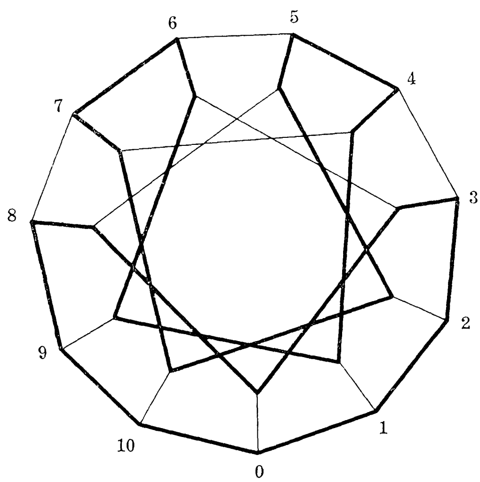

FIGURE 1

Clearly H is a 2-factor, and $\varphi(H) = H$. If C_0 is the circuit of H which contains 0, then $\varphi(C_0) = C_0$. If C_0 has odd length, then it must contain an odd number of edges of the form $(i, -i)$ and $(i', -i')$. The only candidates are:

(A) $(m, m + 1)$

(B) $\left(\left(n - \frac{k}{2}\right)', \left(\frac{k}{2}\right)'\right)$

(C) $\left(\left(\frac{n - k}{2}\right)', \left(\frac{n + k}{2}\right)'\right)$.

The edge (C) is not in H by our construction. Either the presence of (A) in H or the existence of edge (B) will imply that k is even. We conclude that if k is odd C_0 has even length.

Let $m - k + 2 \leqq i \leqq m + k - 1$. Then either $i', i, i + 1, (i + 1)'$ or $i', i, i - 1, (i - 1)'$ are 4 consecutive vertices on a circuit of H. We call such sets 4-sets. If every point of a circuit is on a 4-set, then the circuit has even length.

Now consider a vertex i', $m + k < i \leqq n - 1$ or $0 \leqq i < m - k + 1$, which is not on C_0. The circuit of H which contains i' passes consecutively through the the vertices $i', (i + k)', (i + 2k)' \cdots (i + rk)'$, $(i + (r + 1)k)'$, where $i + rk < m - k + 1, i + (r + 1)k > m - k + 1$, $r \geqq 0$. The vertex $(i + (r + 1)k)'$ is on a 4-set, and also $i + (r + 1)k \leqq m$, hence the circuit continues through the vertices $i + (r + 1)k, i + (r + 1)k \pm 1, (i + (r + 1)k \pm 1)', (i + rk \pm 1)' \cdots (i \pm 1)'$. The circuit continues to $(i \pm 1 - k)'$ and by an identical argument eventually returns and hits i' or $(i + 2)'$ or $(i - 2)'$. In the first case the circuit is complete and it is easily seen that it has even length. The other two cases lead to a contradiction; for assume (w.l.o.g) that the circuit is on $(i', (i + 1)', (i + 2)')$. Then by the above argument the circuit will eventually hit either $(i + 1)'$ again or else $(i + 3)'$. But the first case is impossible, because H is bivalent. Hence the circuit contains $(i + 3)'$ and further $(i + 4)' \cdots (m - k + 1)'$, but this contradicts our assumption, as $(m - k + 1)'$ is on C_0.

CONSTRUCTION 2. H has the following edges:

(a) On the outer rim: $(n - 1, 0), (0, 1), (2, 3), \cdots, (2j, 2j + 1) \cdots (n - 3, n - 2)$.

(b) Spokes: all, except $(0, 0')$.

(c) On the inner rim: $(0', k'), (2k', 3k'), \cdots (2jk', (2j + 1)k'), \cdots, ((n - 1)k', 0')$.

(For the sake of clarity we have written ck' instead of the formally more correct $(ck)'$.)

EXAMPLE. $G(15, 4)$. See Figure 2.

Again, one checks easily that H is a 2-factor and that $\varphi(H) = H$. Looking at the edges (A), (B), and (C) of Construction 1, we note that (C) is not an edge if k is even. If edge (A) occurs, then $m = (n - 1/)2$ is even and $n \equiv 1 (\text{mod } 4)$. If edge (B) occurs, and we write $k/2 \equiv jk (\text{mod } n), j < n$, then j is odd by our construction. But then $k \equiv 2jk (\text{mod } n) \Rightarrow (2j - 1) \equiv 0 \ (\text{mod } n) \Rightarrow n = 2j - 1 \Rightarrow n \equiv 1 \ (\text{mod } 4)$.

Hence if $n \equiv 3 (\text{mod } 4)$ and k is even none of the lines $(A), (B)$, and (C) occur, and we may conclude by the argument used in Construction 1 that the circuits through 0 and $0'$ have even length. All

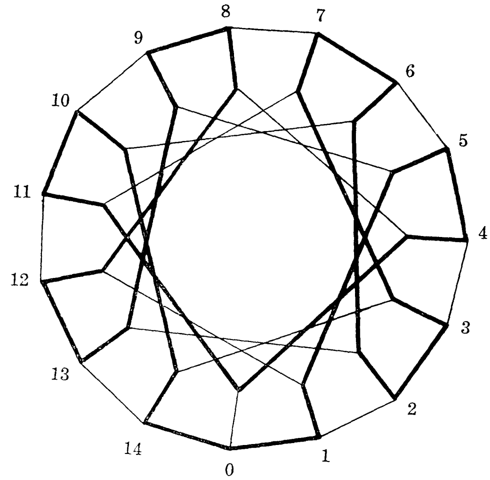

FIGURE 2

the points of every other circuit belong to a 4-set, and hence also have even length. Therefore H if a Tait cycle if $n \equiv 3(\text{mod } 4)$ and k is even.

If $n \equiv 1(\text{mod } 4)$ and k and k^{-1} are both even, then the edge $((k+1)', 1') = (1', (k+1)') = (k^{-1}k', (k^{-1}+1)k')$ exists in H, and so does the edge $(-1', -(k+1)')$. We then obtain the circuit:

$$0', k', k, k+1, (k+1)', 1', 1, 0, -1, -1',$$
$$-(k+1)', -(k+1), -k, -k', 0'$$

which has length 14 and contains both 0 and $0'$.

We conclude that in this case H is again a Tait cycle.

CONSTRUCTION 3. For this construction we assume $n \equiv 1(\text{mod } 4)$, k even, k^{-1} odd and $> n/2$. This last assumption is no real restriction, because if k^{-1} is odd and $< n/2$, then Construction 1 gives a Tait cycle for $G(n, k^{-1})$ and Watkins has shown that $G(n, k)$ and $G(n, k^{-1})$ are isomorphic. Finally we need to assume $k > 2$; this restriction was not needed in Constructions 1 and 2.

H has the following edges:

On the outer rim: $(-1, 0), (0, 1), (2, 3), \cdots, (k-4, k-3), (k-2, k-1), (k-1, k), (k+1, k+2), \cdots (n-k-2, n-k-1), (n-k, n-k+1), (n-k+1, n-k+2), (n-k+3, n-k+4), \cdots, (n-3, n-2)$.

Spokes: all except $(0'0')$, $(k-1, (k-1)')$, $(n-k+1, (n-k+1)')$.

On the inner rim: $(0', k'), (2k', 3k'), \cdots, ((n-k^{-1})k', (n-k^{-1}+1)k'), ((n-k^{-1}+1)k', (n-k^{-1}+2)k'), ((n-k^{-1}+3)k', (n-k^{-1}+4)k')), \cdots, ((k^{-1}-2)k', (k^{-1}-1)k'), ((k^{-1}-1)k', k^{-1}k'), ((k^{-1}+1)k', (k^{-1}+2)k'), \cdots, ((n-1)k', 0')$.

EXAMPLE. $G(17, 4)$

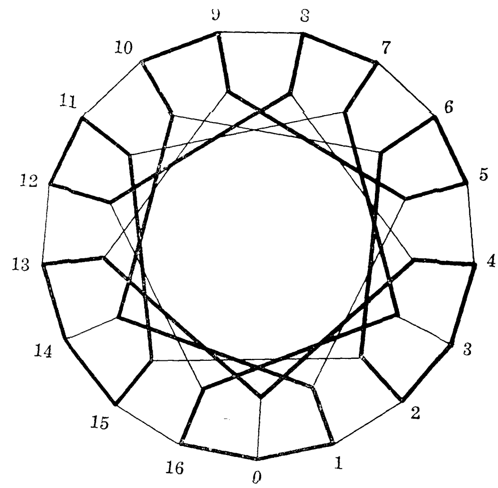

FIGURE 3

H is a 2-factor, as long as $n - k^{-1} + 1 < k^{-1} - 1$, which assures that the constructed edges on the inner rim cover all vertices of the inner rim. But this condition holds whenever $k^{-1} > (n + 1/2)$ or alternatively when $k^{-1} > (n/2)$, and $k > 2$. It is clear that $\varphi(H) = H$.

Since $n \equiv 1(\text{mod } 4)$, m is even and $(m, m + 1)$ is not an edge of H. As $(n - k/)2$ is not an integer H does not have an edge $((n - k)/2)'$, $(n + k)/2)')$. Finally, since $n - k^{-1} + 1 \leqq (n - 1)/2 = m < m + 1 = (n + 1)/2 \leqq k^{-1} - 1$, and m is even, H does not contain the edge $(mk', (m + 1)k') = (-k'/2, k'/2)$. As before we conclude that the circuits containing 0 and $0'$ have even length. The circuit containing 0 also contains $n - 1$, $(n - 1)'$, $(k - 1)'$ and $1, 1'$, $(n - k + 1)'$, while the circuit containing $0'$ also contains k', k, $k - 1$, $k - 2$, $(k - 2)'$ and $(n - k)'$, $n - k$, $n - k + 1$, $n - k + 2$, $(n - k + 2)'$. Hence the other circuits only contain vertices of 4-sets and every circuit of H has even length.

We note that our constructions are not mutually exclusive. For example, Construction 1 also produces a Tait cycle, when k is even, and the largest positive integer q such that $qk < n$ is an odd number.

We conclude with a new conjecture. G. N. Robertson [1] has shown that $G(n, 2)$ is Hamiltonian unless $n \equiv 5(\text{mod } 6)$. As $G(n, 2) \cong G(n, (n + 1)/2) \cong G(n, (n - 1)/2) \cong G(n, n - 2)$ (see [2]), none of these graphs has a Hamiltonian if $n \equiv 5(\text{mod } 6)$. We conjecture that all other generalized Petersen graphs are Hamiltonian. In all examples that we have worked out $G(n, k)$ possesses a Hamiltonian H with $\varphi(H) = H$, but our three constructions are Hamiltonians only in a minority of cases.

References

1. G. N. Robertson, *Graphs under Girth, Valency, and Connectivity Constraints* (Dissertation), University of Waterloo, Waterloo, Ontario, Canada, 1968.
2. Mark E. Watkins, *A theorem on Tait colorings with an application to the generalized Petersen graphs*, J. Combinatorial Theory, **6** (1969), 152-164.

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