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ASYMPTOTICS FOR A CLASS OF WEIGHTED EIGENVALUE PROBLEMS

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Abstract: This paper deals with the asymptotic behavior at infinity of the solutions to $\ell(y) = \lambda wy$ on $[a, \infty)$ where ℓ is an nth order ordinary linear differential operator, λ is a nonzero complex number and w is a suitably chosen positive valued continuous functions. As an application the deficiency indices of certain symmetric differential operators in Hilbert space are computed.

1. Preliminaries. Throughout the first three sections ℓ will denote an operator of the form,

$$\ell(y) = y^{(n)} + \sum_{k=2}^{n} p_k y^{(n-k)} \quad \text{on } [a, \infty)\ , \tag{1.1}$$

where each of $p_2, \cdots, p_n$ is a continuous complex valued function on $[a, \infty)$. In view of the transformation indicated on p. 309 of [2] it results in no great loss of generality to take the coefficient of $y^{(n-1)}$ to be zero, and in order to simplify the exposition we shall do this. We shall be concerned with the behavior at infinity of the solutions to

$$\ell(y) = \lambda wy \quad \text{on } [a, \infty) \tag{1.2}$$

where λ is a nonzero complex number and w is an appropriate weight (i.e., positive valued continuous function). For a given ℓ we shall consider the weights w indicated by the following definition. $\mathscr{L}(a, \infty)$ denotes the Banach space of all complex valued measurable functions which are absolutely Lebesgue integrable on $[a, \infty)$.

DEFINITION. If ℓ is as in 1.1 the statement that w is an ℓ-admissible weight means that

(1) w is differentiable, strictly increasing, and unbounded on $[a, \infty)$;

(2) each of $[w'/w^{1+1/n}]'$ and $[(w'/w)^2(1/w^{1/n})]$ is continuous on $[a, \infty)$ and is in $\mathscr{L}(a, \infty)$; and

(3) $p_j/w^{(j-1)/n} \in \mathscr{L}(a, \infty)$ for $j = 2, 3, \cdots, n$.

For example if $\ell(y)(t) = y''(t) \pm t^{\alpha}y(t)$ for $t \geqq 1$ and $w(t) = t^{\beta}$ then w will be an ℓ-admissible weight if and only if $\beta > 0$ and $\beta > 2(\alpha + 1)$.

We shall demonstrate that when w is an ℓ-admissible weight the solutions of 1.2 have a particularly simple asymptotic behavior and we shall establish that every operator of the form 1.1 has admissible weights.

Our asymptotic theorem relies on the classic perturbation theorem of Norman Levinson [2, Therem 8.1 p. 92 or 10]. Recent related works include [3, 7, 8, 9, 11, and 12]. The results in § 4 complement those of reference [13].

2. Results. Our main results are stated in the following two theorems.

THEOREM 1. *If ℓ is as in 1.1 and U is a continuous function on $[a, \infty)$ there is an ℓ-admissible weight w with $w(t) \geqq U(t)$ for $t \geqq a$.*

THEOREM 2. *If ℓ is as in 1.1, w is an ℓ-admissible weight, and λ is a nonzero complex number then equation 1.2 has n linearly independent solutions $y_1, \cdots, y_n$ such that for $k = 0, \cdots, n-1$*

$$y_j^{(k)}(t) w^{\alpha_k}(t) e^{-\mu_j h(t)} \longrightarrow \mu_j^k \text{ as } t \longrightarrow \infty ,$$

where

$$h(t) = \int_a^t w^{1/n} ,$$

$\mu_1, \cdots, \mu_n$ are the distinct nth roots of λ, and $\alpha_{k-1} = (n - 2k + 1)/2n$ for $k = 1, \cdots, n$.

3. Proofs. The proof of Theorem 1 will be facilitated by the following results.

LEMMA. *If $r > 1$ and $1 < c < d$ there exist positive constants M_r and N_r, depending only on r, and a function f defined on $[0, 1]$ such that*

(1) *f is continuously differentiable, strictly increasing, $f(0) = c$, $f(1) = d$, and $f'(0) = 0 = f'(1)$;*

(2) *$[f'/f^r]'$ exists and is continuous on $[0, 1]$ and has the value 0 at 0 and at 1; and*

(3) *$|[f'/f^r]'(x)| \leqq M_r c^{1-r}$ and $[(f'/f)^2 f^{1-r}](x) \leqq N_r c^{1-r}$ for all $x \in [0, 1]$.*

Proof. Given $r > 1$ and $1 < c < d$ let g: $[0, 1] \to [0, 1]$ be a twice continuously differentiable fuction such that $g(0) = 0$, $g(1) = 1$, $g'(x) > 0$ for $x \in (0, 1)$, and $g'(0) = g''(0) = g'(1) = g''(1) = 0$ (e.g. let $g(x) = h(h(x))$ where $h(x) = (2x - x^2)^2$). Then let f: $[0, 1] \to [c, d]$ be given by

$$f = \left\{c^{1-r} - 6(c^{1-r} - d^{1-r})\left[\left(\frac{1}{2}\right)g^2 - \left(\frac{1}{3}\right)g^3\right]\right\}^{1/(1-r)} ,$$

clearly $f(0) = c$ and $f(1) = d$. Since each of g and the function whose value at x is $(1/2)x^2 - (1/3)x^3$ is strictly increasing on $[0, 1]$ and since $1 - r < 0$ and $1 < c < d$ we see that f is strictly increasing on $[0, 1]$. Using the above listed properties of g we see that f' is continuous on $[0, 1]$ and that $f'(0) = 0 = f'(1)$. Computation shows that

$$[f'/f^r] = (6/(r-1))(c^{1-r} - d^{1-r})(g - g^2)g'$$

and

$$[f'/f^r]' = (6/(r-1))(c^{1-r} - d^{1-r})[(1-2g)(g')^2 + (g - g^2)g''] \, .$$

Hence condition (2) of the lemma is satisfied. Letting M_r be a bound for $(6/(r-1))[(1-2g)(g')^2 + (g-g^2)g'']$ on $[0, 1]$ we see that

$$|[f'/f^r]'(x)| \leqq M_r c^{1-r} \quad \text{for } x \in [0, 1] \, .$$

Noting that $c^{1-r} \geqq (f(x))^{1-r} \geqq d^{1-r}$ for $x \in [0, 1]$ and letting N_r be a bound for $[(6/(r-1))(g - g^2)g']^2$ on $[0, 1]$ we see that

$$|[(f'/f^r)^2 f^{1-r}](x)| \leqq N_r c^{3(1-r)} \leqq N_r c^{1-r}$$

for $x \in [0, 1]$, and the lemma is proved.

Proof of Theorem 1. We shall make use of the fact that if U is a continuous function on $[a, \infty)$ and γ is a positive number there is a weight w such that $U/w^\gamma \in \mathscr{L}(a, \infty)$. To see this let w be such that

$$\frac{1 + |U(t)|}{w^\gamma(t)} = \frac{1}{(t - a + 1)^2} \, .$$

Given an ℓ as in 1.1 and a continuous function U on $[a, \infty)$ choose weights $v_2, v_3, \cdots, v_n$ such that $p_j/v_j^{(j-1)/n} \in \mathscr{L}(a, \infty)$ for $j = 2, \cdots, n$ and let v be a weight such that $v(t) \geqq \max\{U(t), v_2(t), \cdots, v_n(t)\}$ for all $t \geqq a$. Next let $\{c_k\}_{k=1}^\infty$ be a strictly increasing sequence of numbers with $c_k \geqq k^{2n}$ and $c_k \geqq$ maximum of $v(t)$ for $t \in [a+k-1, a+k]$ and let f_k be a function satisfying the conclusion to the lemma with $r = 1 + 1/n$, $c = c_k$ and $d = c_{k+1}$ for each k. Let w be defined by

$$w(t) = f_k(t - a - k + 1) \quad \text{for} \quad t \in [a + k - 1, \, a + k] \, .$$

Clearly then w satisfies condition (1) in the definition of admissible weight, and since $w(t) \geqq v(t)$, we see that $w(t) \geqq U(t)$ and w satisfies condition (3) of the definition. To see that condition (2) is satisfied note that

$$\int_a^\infty |[w'/w^{1+1/n}]'| = \sum_{k=1}^\infty \int_0^1 |[f_k'/f_k^{1+1/n}]'|$$
$$\leqq \sum_{k-1}^\infty M_{1+1/n} c_k^{-1/n} \leqq M_{1+1/n} \sum_{k=1}^\infty k^{-2} < \infty ,$$

and

$$\int_a^\infty [w'/w)^2(1/w^{1/n})] = \sum_{k=1}^\infty \int_0^1 [(f_k'/f_k)^2 f_k^{-1/n}]$$
$$\leqq \sum_{k=1}^\infty N_{1+1/n} c_k^{-1/n} \leqq N_{1+1/n} \sum_{k=1}^\infty k^{-2} < \infty .$$

Proof of Theorem 2. We shall establish the theorem by showing that the standard vector-matrix formulation,

$$y' = \begin{bmatrix} 0 & 1 & 0 \cdots 0 & 0 \\ 0 & 0 & 1 \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots 0 & 1 \\ (\lambda w - p_n) & -p_{n-1} & -p_{n-2} \cdots -p_2 & 0 \end{bmatrix} y \tag{3.1}$$

of equation (1.2) has a fundamental matrix Y_0 such that

$$Q(t)Y_0(t)E(t) \longrightarrow L \quad \text{as} \quad t \longrightarrow \infty ,$$

where

$$Q = \text{diag}\,[w^{\alpha_1}, \cdots, w^{\alpha_n}]$$

with $\alpha_k = (n - 2k + 1)/2n$ for $k = 1, \cdots, n$;

$$E(t) = \text{diag}\,[e^{-\mu_1 h(t)}, \cdots, e^{-\mu_n h(t)}]$$

with $\mu_1, \cdots, \mu_n$ the distinct nth roots of λ and

$$h(t) = \int_a^t w^{1/n} ;$$

and

$$L = \begin{bmatrix} 1 & 1 \cdots 1 \\ \mu_1 & \mu_2 \cdots \mu_n \\ \mu_1^2 & \mu_2^2 \cdots \mu_n^2 \\ \cdots & \cdots \\ \mu_1^{n-1} & \mu_2^{n-1} \cdot \mu_n^{n-1} \end{bmatrix} .$$

Using this notation we begin by letting Y be a fundamental matrix for equation (3.1). Since h is strictly increasing on $[a, \infty)$ we may let g be the function inverse to it ($h(g(s)) = s$ for $s \geqq 0$) and let $Z(s) = Q(g(s))\ Y(g(s))$ for $s \geqq 0$. Noting that $g'(s) = 1/h'(g(s))$ and that $Q(g(s))$ is nonsingular we see that Z is a fundamental matrix for

$$z'(s) = [1/h'(g(s))] \tag{3.2}$$
$$[Q(g(s)M(g(s))Q^{-1}(g(s)) + Q'(g(s))Q^{-1}(g(s))]z(s)$$

where M is the coefficient matrix on the right hand side of equation (3.1). Computation shows that equation (3.2) is the same as

$$z'(s) = [A + \alpha(s)D + R(s)]z(s), \ s \geqq 0 \tag{3.3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \lambda & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (n \times n),$$

$$\alpha(s) = [w'/w^{1/1n}](g(s)) ,$$

$$D = \operatorname{diag}[\alpha_1, \cdots, \alpha_n] ,$$

and $R(s)$ is the $n \times n$ matrix having

$$[(-p_j/w^{(j-1)/n})(1/w^{1/n})](g(s))$$

as its $n, n-j+1$ entry for $2 \leqq j \leqq n$ and zero for all other entries. Since $h(t) \to \infty$ as $t \to \infty$ and

$$\int_0^{h(b)} |[(p_j/w^{(j-1)/n})(1/w^{1/n})](g(s))|\,ds = \int_a^b |[(p_j/w^{(j-1)/n})](t)|\,dt$$

we see from condition (3) of the definition of $\mathscr{L}$-admissible weight that $|R| \in \mathscr{L}(0, \infty)$ since by condition (2) of the definition it is the case that $[w'/w^{1+1/n}]'$ and $[w'/w)^2(1/w^{1/n})]$ are in $\mathscr{L}(a, \infty)$ we see from similar "changes of variable" that α' and α^2 are in $\mathscr{L}(0, \infty)$. Since $\alpha' \in \mathscr{L}(a, \infty)$ and α' is continuous, α has a limit at ∞ and since $\alpha^2 \in \mathscr{L}(0, \infty)$ this limit must be zero. The characteristic roots of A are $\mu_1, \cdots, \mu_n$. Hence for $j = 1, \cdots, n$ we may let λ_j be the continuous function such that $\lambda_j(s) \to \mu_j$ as $s \to \infty$ and $\lambda_j(s)$ is a characteristic root of $A + \alpha(s)D$ for $s \geqq 0$.

We now shall show that $\lambda_j - \mu_j \in \mathscr{L}(0, \infty)$ for each j. Following the procedure used in [9] we note that

$$\begin{aligned} 0 &= \det[A + \alpha(s)D - \lambda_j(s)I] \\ &= (-1)^{n+1}\lambda + \pi_{i=1}^n(\alpha_i\alpha(s) - \lambda_j(s)) \\ &= (-1)^{n+1}\lambda + (-\lambda_j(s))^n + (-\lambda_j(s))^{n-1}\alpha(s)\sum_{i=1}^n \alpha_i + \alpha^2(s)F(s) , \end{aligned}$$

where F is a bounded function. (Racall that $\alpha(s) \to 0$ and $\lambda_j(s) \to \mu_j$ as $s \to \infty$.) Noting that $\sum_{i=1}^n \alpha_i = 0$ we then have

$$0 = (-1)^{n+1}\lambda + (-\lambda_j(s))^n + \alpha^2(s)F(s)$$

and

$$0 = (-1)^{n+1}\lambda + (-\mu_j)^n .$$

From which we conclude that

$$(\lambda_j(s))^n - \mu_j^n = -(-1)^n \alpha^2(s)F(s)$$

and

$$|\lambda_j(s) - \mu_i| \, |\textstyle\sum_{i=1}^n (\lambda_j(s))^{n-i}\mu_j^{i-1}| \leqq |\alpha^2(s)F(s)| .$$

Since $\lambda_j(s) \to \mu_j \neq 0$ as $s \to \infty$, since $\alpha^2 \in \mathscr{L}(a, \infty)$ and since F is bounded we see that $|\lambda_j(s) - \mu_j|$ is for all large s dominated by a function in $\mathscr{L}(0, \infty)$; hence $\lambda_j - \mu_j \in \mathscr{L}(0, \infty)$.

Thus all the hypotheses of Thorem 8.1 p. 92 of [2] are satisfied and noting that the jth column of L is an eigenvector of A corresponding to μ_j we are able to conclude that there exist numbers $s_1, \cdots, s_n$ and a fundamental matrix Z_0 for equation (3.3) such that

$$Z_0(s)G(s) \longrightarrow L \quad \text{as} \quad s \longrightarrow \infty$$

where

$$G(s) = \exp\left\{\operatorname{diag}\left[-\int_{s_1}^s \lambda_1, \cdots, -\int_{s_n}^s \lambda_n\right]\right\} .$$

Since $\lambda_j - \mu_j \in \mathscr{L}(a, \infty)$ it follows that there is a nonsingular diagonal constant matrix H such that

$$Z_0(s)H \operatorname{diag}[e^{-\mu_1 s}, \cdots, e^{-\mu_n s}] \longrightarrow L \quad \text{as} \quad s \longrightarrow \infty .$$

(See the procedure followed at the end of the proof of Theorem 2.3 in [12].) Since each of Z_0H and Z is a fundamental matrix for equation (3.3) there is a constant nonsingular matrix C such that $Z_0H = ZC$. Letting Y_0 be YC and recalling that $Z(s) = Q(g(s))Y(g(s))$ we have

$$Q(g(s))Y_0(g(s)) \operatorname{diag}[e^{-\mu_1 s}, \cdots, e^{-\mu_n s}] \longrightarrow L \quad \text{as} \quad s \longrightarrow \infty .$$

Hence $Q(t)Y_0(t)E(t) \to L$ as $t \to \infty$ and the theorem is proved.

4. Application.

If w is a weight on $[a, \infty)$ we denote by $\mathscr{L}^2(w\colon a, \infty)$ the Hilbert space of all complex valued measurable y such that

$$\int_a^\infty |y|^2 w < \infty$$

with the obvious inner product. If ℓ is an nth formally self-adjoint (in the sense defined in [2]; see in particular 13 and 14 p. 204) operator, w is a weight,

$$\mathscr{D} = \{y \mid y \in \mathscr{L}^2(w\colon a, \infty),\ y^{(n-1)} \text{ is absolutely continuous and } (1/w)\ \ell(y) \in \mathscr{L}^2(w\colon a, \infty)\}\,,$$

$$\mathscr{D}_0' = \{y \mid y \in \mathscr{D} \text{ and has compact support interior to } [a, \infty)\}\,.$$

and L and L_0' are the restriction of $(1/w)\ \ell$ to $\mathscr{D}$ and $\mathscr{D}_0'$ respectively then L_0' is a densely defined symmetric operator in $\mathscr{L}^2(w\colon a, \infty)$, hence admits a closure L_0 in this space, and $L_0^* = L$ where $*$ denotes adjoint operator in $\mathscr{L}^2(w\colon a, \infty)$. Verification of these assertions closely parallels that for the case $w \equiv 1$ found in [1], [4], and [11].

The deficiency indices of L_0 are (n_1, n_2) where n_j is the dimension of the subspace of solutions to

$$\ell(y) = (-1)^{j+1} i w y$$

which lie in $\mathscr{L}^2(w\colon a, \infty)$. (Actually for ℓ formally self-adjoint any λ in the upper half plane may be used for i and any λ in the lower half plane for $-$i. See [4 Theorem 19, p. 1232, 5, and 6].)

By use of Theorem 2 we may conclude the following.

THEOREM 3. *Let ℓ be as in* 1.1 *and let w be an ℓ-admissible weight.*

(1) *If n is even and* $\operatorname{Im}\lambda \neq 0$ *the dimension of the subspace of solution to equation* 1.2 *which lie in* $\mathscr{L}^2(w\colon a, \infty)$ *is* $n/2$.

(2) *If* $n = 4k + 1 = 2m + 1$ *and* $\operatorname{Re}\lambda > 0$ *or if* $n = 4k + 3 = 2m + 1$, *and* $\operatorname{Re}\lambda < 0$ *the dimension of the subspace is m.*

(3) *If* $n = 4k + 1 = 2m + 1$, *and* $\operatorname{Re}\lambda < 0$ *or if* $n = 4k + 3 = 2m + 1$, *and* $\operatorname{Re}\lambda > 0$ *the dimension is* $m + 1$.

Proof. We begin by noting that for c real, w an ℓ-admissible weight for some ℓ, and $E \subset [a, \infty)$ with E of infinite Lebesgue measure (for the first application below we will take $E = [a, \infty)$),

$$\int_E \exp\left\{\int_a^t w^{1/n}[c + (1/n)(w'/w^{1+1/n})]\right\} dt\,, \tag{4.1}$$

is finite if $c < 0$ and infinite if $c > 0$. To see this recall that in the proof of Theorem 2 we showed that $\alpha(s) = [w'/w^{1+1/n}](g(s)) \to 0$ as $s \to \infty$. Hence $[w'/w^{1+1/n}](t) = \alpha(h(t)) \to 0$ as $t \to \infty$. Since $w(t) \to \infty$ as $t \to \infty$ we then see that $w^{1/n}(t)[c + (1/n)(w'/w^{1+1/n})(t)] > c$ for $c > 0$ and $< c$ for $c < 0$ for all large t and the above assertion is immediate.

We next observe from Theorem 2 that if w is an ℓ-admissible weight then equation 1.2 has n lineary independent solutions $U_1, \cdots, U_n$ (with $U_j = (w(a))^{(n-1)/2n} y_j$) such that

$$(4.2) \qquad |U_j(t)|^2 w(t) = (1 + o(1)) \exp\left\{ \int_a^t w^{1/n}[2\,\mathrm{Re}\mu_j + (1/n)(w'/w^{1+1/n})]\right\} .$$

If $n = 2m$ and $\mathrm{Im}\,\lambda \neq 0$ we may arrange the nth roots of λ so that

$$\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2 < \cdots < \mathrm{Re}\mu_m < 0 < \mathrm{Re}\mu_{m-1} < \cdots < \mathrm{Re}\mu_n .$$

Thus each of $U_1, \cdots, U_m$ will lie in $\mathscr{L}^2(w\colon a, \infty)$; and if $c_{m+1}, c_{m+2}, \cdots, c_n$ are not all zero and j is the largest integer with $m + 1 \leqq j \leqq n$ such that $c_j \neq 0$ then

$$\sum_{k=1}^{m} c_{m+k} U_{m+k} = c_j U_j (1 + o(1)) \notin \mathscr{L}^2(w\colon a, \infty) .$$

Hence the first assertion of the theorem is established.

In case $\mathrm{Im}\ \lambda \neq 0$ the last two assertions follow analogously upon noting that in Case 2 if $\mathrm{Im}\,\lambda \neq 0$ the n-th roots may be arranged so that

$$\mathrm{Re}\mu_1 < \cdots < \mathrm{Re}\mu_m < 0 < \mathrm{Re}\mu_{m+1} < \cdots < \mathrm{Re}\mu_n ,$$

and that in Case 3 they may be arranged so that

$$\mathrm{Re}\mu_1 < \cdots < \mathrm{Re}\mu_{m+1} < 0 < \mathrm{Re}\mu_{m+2} < \cdots < \mathrm{Re}\mu_m .$$

If λ is real and positive and $n = 4k + 1$ the roots may be arranged so that

$$\begin{aligned} \mathrm{Re}\mu_1 &= \mathrm{Re}\mu_2 < \mathrm{Re}\mu_3 \\ &= \mathrm{Re}\mu_4 < \cdots < \mathrm{Re}\mu_{2k-1} \\ &= \mathrm{Re}\mu_{2k} < 0 < \mathrm{Re}\mu_{2k+1} \\ &= \mathrm{Re}\mu_{2h+2} < \cdots < \mathrm{Re}\mu_{n-2} \\ &= \mathrm{Re}\mu_{n-1} < \mathrm{Re}\mu_n , \end{aligned}$$

and so that if $\mu_j = \mu_{j+1}$ then $\mathrm{Im}\,\mu_{j+1} > 0$. Then each of $U_1, \cdots, U_{2k}$ is in $\mathscr{L}(a, \infty)$, and each of $U_{2k+1}, \cdots, U_n$ is not in $\mathscr{L}^2(a, \infty)$. It remains to be shown that no nontrivial linear combination of $U_{2k+1}, \cdots, U_n$ lies in $\mathscr{L}^2(a, \infty)$ and to do this it is sufficient to show if $2k + 1 \leqq j < n$ with j odd then no nontrivial linear combination of U_j and U_{j+1} lies in $\mathscr{L}^2(a, \infty)$.

Suppose that $c_1 U_j + c_2 U_{j+1} \in \mathscr{L}^2(w\colon a, \infty)$ with c_1 and c_2 not both zero and j odd with $2k + 1 \leqq j < n$. Since $U_j \notin \mathscr{L}^2(w\colon a, \infty)$, it follows that $c_1 \neq 0$ and $U_j + c U_{j+1} \in \mathscr{L}^2(w\colon a, \infty)$ where $c = c_2/c_1$. From Theorem 2 and the definition of $U_1, \cdots, U_n$ we have that

$U_j(t) + cU_{j+1}(t) = U_j(t)\,[1 + c(U_{j+1}(t)/U_j(t))]$ is

$$(4.3)\qquad (1 + o(1))\,U_j(t)\left\{1 + (c + o(1))\exp\left[\int_d^t 2i\,(\operatorname{Im}\mu_{j+1})w^{1/n}\right]\right\}.$$

For all large t. Hence $|c| = 1$ for if $|c| \neq 1$ the term in { } would be bounded away from zero for all large t and this would contradict the fact that $U_j \notin \mathscr{L}^2(w\colon a, \infty)$. Letting $E = \{t \mid$ modulus of term in 4.3 in { } is $\geqq \sqrt{2}\}$ we see since w is increasing that E is of infinite measure. (Think of the exponential term in 4.3 or giving the position of a particle on the unit circle at time t moving counterclockwise at an ever increasing rate.) Hence from 4.3 we see that for some constant K,

$$\int_E |U_j|^2 w \leqq K\int_a^\infty |U_j + cU_{j+1}|^2 w < \infty\ .$$

But from 4.1 and 4.2 we see that

$$\int_E |U_j|^2 w = \infty$$

must be the case. This contradiction shows then that $c_1U_j + c_2U_{j+1} \notin \mathscr{L}^2(w\colon a, \infty)$.

The proofs of the remaining assertions when λ is real are naalogous.

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