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INTERPOLATION BY ANALYTIC FUNCTIONS

ARNE STRAY

It is shown that interpolation problems for $R(X)$, $A(X)$ and $H^\infty(X^\circ)$ are local problems whenever X is a compact plane set.

Introduction and notation. Let X be compact plane set, X° its interior and $\partial X = X \backslash X^\circ$ its boundary.

$H^\infty(X^0)$ denotes all bounded complex-valued analytic functions on X^0. $A(X)$ is all continuous functions on X which are analytic in X^0. $R(X)$ denotes the uniform closure on X of the rational functions with poles outside X.

A subset E of X is an interpolation set for $A(X)$ if $A(X)\backslash E$ (the restrictions to E of the functions in $A(X)$) equals the space $C(E)$ of all continuous complex-valued functions on E.

E is called a peak set for $A(X)$ if there exists $f \in A(X)$ such that $f = 1$ on E and $|f(x)| < 1$ if $x \in X \backslash E$.

A peak interpolation set for $A(X)$ is a set E which has both these properties. Peak and interpolation sets for $R(X)$ are defined in the same way.

A sequence $S = \{z_n\}$ of distinct points is called an interpolating sequence for $H^\infty(X^0)$ if for any bounded sequence $\{w_n\}$ of complex numbers there exists $f \in H^\infty(X^0)$ such that $f(z_n) = w_n$ for each n. (For more about interpolating sequences see Ch. 10 in [3].)

If F is a subset of the complex plane we give it (as a toplogical space) the topology induced from $\boldsymbol{C}$. $C_b(F)$ is the Banach space of all bounded continuous complex-valued functions on F. We also consider $H^\infty(X^0)$, $R(X)$ and $A(X)$ as Banachspaces with the usual sup norm.

Let us mention two other Banach-spaces of analytic functions which has not been much studied yet, but which may be useful in characterizing interpolation sets for $R(X)$ and $A(X)$ among other things.

$HR(X)$ denotes all functions on X^0 which are pointwise limits on X^0 of bounded sequences in $R(X)$. For each $f \in HR(X)$ we define

$$||f||_{HR} = \inf \{\sup_n ||f_n||: \{f_n\} \subset R(X),\ f_n \longrightarrow f \text{ pointwise on } X^0\} .$$

With this norm $HR(X)$ clearly is a Banach space. In the same way we define $HA(X)$ corresponding to $A(X)$ and it is also a Banach space with the norm $||\ \ ||_{HA}$. Very recently A. M. Davie has shown that the norm $||\ \ ||_{HA}$ is the same as sup norm on X^0 and the same is proved for $||\ \ ||_{HR}$ if almost every point of ∂X (w.r.t. area) is a peak point for $R(X)$. We shall not need these interesting results here. (See [1] for his results.) Some results about $HR(X)$ can be found in [2].

If f is a complex-valued function defined on a set F and $S \subset F$ is a subset we define $\|f\|_S$ as $\sup\{|f(z)|: z \in S\}$.

A typical problem we shall study in this paper is the following:

Let S be a sequence in X^0. What local conditions on S are sufficient to conclude that S is an interpolating sequence for $H^\infty(X^0)$?

An obvious necessary condition is that $S \cap \Delta_z$ is an interpolating sequence for $H^\infty(X^0)$ whenever Δ_z is an open disc centered at z for which $\Delta_z \cap S \neq \emptyset$.

Suppose that the following weaker condition is satisfied:

(*): For every $z \in \bar{S}$ (the closure of S) there exists $\delta_z > 0$ such that $S \cap \Delta_z$ is an interpolating sequence for $H^\infty(\Delta_z \cap X^0)$ where $\Delta_z = \{w: |w - z| < \delta^z\}$.

We shall then by definition say that S admits local H^∞-interpolation w.r.t. X^0.

Our main result is the following:

THEOREM 1. *Let X be a compact set with nonempty interior X^0. A sequence S in X^0 is an interpolating sequence for $H^\infty(X^0)$ if and only if S admits local H^∞-interpolation w.r.t. X^0.*

Some time after Theorem 1 was proved we learnt about a result of J. Rainwater which has some connection with Theorem 1. If in the definition of local H^∞-interpolation the condition ($*$) had been replaced by the other necessary condition for interpolation mentioned above Theorem 1 would be a somewhat weaker result.

We want to point out this weaker result is easy to deduce from J. Rainwaters paper. (See [4].) We also want to point out that a theorem of E. L. Stout on interpolating sequences in multiply connected domains in an easy consequence of Theorem 1. (See [5].)

Interpolating sequences can clearly also be defined for $HR(X)$ and $HA(X)$. It should also be clear what is meant by saying that a sequence $S \subset X^0$ admits local HR-interpolation (or HA-interpolation) w.r.t. X.

It will follow from our proof that Theorem 1 also holds for $HR(X)$ and $HA(X)$. We shall give some reasons for this at the end of the proof.

LEMMA 1. *Let X be as in Theorem 1 and $z_0 \in \partial X$. Let $0 < r_1 < r_2$ and define $0_1 = \{w: |w - z_0| < r_1\}$ and $0_2 = \{w: |w - z_0| > r_2\}$. Suppose there exists $z_1 \in \boldsymbol{C}\backslash X$ such that $r_2 > |z_1 - z_0| > r_1$*

Let S_i be an interpolating sequence for $H^\infty(X^0 \cap 0_i)$ for $i = 1,2$.

Suppose $\bar{S}_i \subset 0_i$ *for* $i = 1, 2$.

Then $S = S_1 \cup S_2$ *is an interpolating sequence for* $H^\infty(X^0)$.

Proof. Put $\Gamma_i = \partial 0_i$ for $i = 1, 2$.

Then $\operatorname{dist}\cdot(S, \Gamma_i) > 0$.

Assume $h \in H^\infty(X^\circ \cap 0_1)$. Extend it to $\boldsymbol{C}$ by defining $h(z) = 0$ if $z \notin X^0 \cap 0_1$.

Let $\delta > 0$ be given. Then cover $\boldsymbol{C}$ by open discs $\Delta_n = \Delta(z_n, \delta)$ (of radius δ and centered at z_n) and choose continuously differentiable functions g_n supported on Δ_n as in the scheme for approximation described on page 210 in [2].

Let T_{g_n} be the integraloperator on $L^\infty(dxdy)$ defined by

$$T_{g_n}(f)(w) = \frac{1}{\pi}\iint \frac{f(w) - f(z)}{w - z}\frac{\partial g_n}{\partial \bar{z}} dxdy$$
$$= f(w)\cdot g_n(w) + \frac{1}{\pi}\iint \frac{f(z)}{z - w}\frac{\partial g_n}{\partial \bar{z}}\, dxdy\,.$$

We mention that $T_{g_n}(f)$ is analytic outside the support of g_n and wherever f is and that $T_{g_n}(f)$ is continuous wherever f is.

Also $f - T_{g_n}(f)$ is analytic in the interior of the set where g_n attains the value 1. (See on p. 28–29 in [2] for more details.)

Put $h_n = T_{g_n}(h)$. We are only interested in those n for which $\bar{\Delta}_n \cap \Gamma_1 \cap X \neq \emptyset$. Assume this happens if and only if $1 \leqq n \leqq N$.

Then $h - \sum_1^N h_n = h - T_{(\Sigma_1^N g_n)}(h)$ is analytic near $\Gamma_1 \cap X$ since $\sum_1^N g_1$ equals 1 near $\Gamma_1 \cap X$.

Now there exist functions $\{H_n\}_{n=1}^N$ analytic outside a compact subset of $D_n = \{w\colon |w - z_n| < 2\delta\}\backslash 0_1$ such that $h_n - H_n$ has a triple zero in the Taylor expansion at infinity and in our situation we can obtain $\|H_n\| \leqq c_1\|h\|$ where c_1 is an absolute constant. (See Theorem 7.4 on p. 213 in [2] and the proof of it.)

Now one has to observe two important facts.

(a) If B is a subset of $\boldsymbol{C}$ and $\operatorname{dist}(B, \Gamma_1 \cap X) > 0$ and $\varepsilon > 0$ one can choose δ depending only on ε and $\operatorname{dist}(B, \Gamma_1 \cap X)$ so small that the sum $f_\delta = h - \sum_1^N (h_n - H_n)$ satisfies

$$\|h - f_\delta\|_B \leqq \varepsilon\|h\|. \tag{1}$$

(b) The functions H_n can be chosen such that its singularities lies on a fixed compact subset of D_n independent of h.

In fact one can find two functions $F_{n,1}$ and $F_{n,2}$ analytic outside a compact subset of D_n such that $\|F_{n,1}\| + \|F_{n,2}\| \leqq 20$ and $H_n = \lambda_{n,1}(h)F_{n,1} + \lambda_{n,2}(h)F_{n,2}$. (See lemma 6.3 on page 209 in [2].)

Here $\lambda_{n,k}(h)$ is a complex number and we have

$$|\lambda_{n,k}(h)| \leqq c_2\|h\| \qquad \text{for } k = 1, 2\,, \tag{2}$$

where c_2 in our situation is an absolute constant. If $F_{n,k}$ is constructed as in the mentioned lemma in [2]. We also mention that the maps $h \to \lambda_{n,k}(h)$ are linear.

(Some details indicating how this can be done, can be found in the proof of Lemma 3.1 in [4].)

Given $\varepsilon > 0$ we first choose δ so small that

$$\|h - f_\delta\|_S < \frac{\varepsilon\|h\|}{4} \tag{3}$$

whenever h is as above. The choose rational functions $r_{n,k}$ with poles only at z_1 such that

$$\sum_{n=1}^{N} (\|F_{n,1} - r_{n,1}\|_{0_1 \cup 0_2} + \|F_{n,2} - r_{n,2}\|_{0_1 \cup 0_2}) < \frac{\varepsilon}{4c_2}\,. \tag{4}$$

Now define A_1: $H^\infty(X^0 \cap 0_1) \to H^\infty(X^0)$ by

$$A_1(h) = [h - \sum_1^N (h_n - (\lambda_{n,1}(h)r_{n,1} + \lambda_{n,2}(h)r_{n,2}))] \,|\, X^0\,.$$

From (1), (2), (3) and (4) we deduce that

(i) $\|A_1(h)\| \leqq c_4\|h\|$ where c_4 depends only on the rational functions $r_{n,k}$.

(ii) $\|A_1(h) - h\|_S \leqq \varepsilon\|h\|/4 + \varepsilon\|h\|/4 = \varepsilon\|h\|/2$.

In addition we also mention that A_1 is linear but this fact will not be needed.

In exactly the same way we define a map A_2: $H^\infty(X^0 \cap 0_2) \to H^\infty(X^0)$.

Suppose now $f \in C_b(S)$. By the open mapping theorem applied to the restriction $H^\infty(0_i \cap X^0) \to C_b(S_i)$ for $i = 1, 2$, there exists a constrat M independent of f and functions $h_i \in H^\infty(0_i \cap X^0)$ such that

$$\|h_i\| \leqq M\|f\| \text{ and } h_i = f \text{ on } S_i \text{ for } i = 1,2\,.$$

Put $h_i = 0$ outside $0_i \cap X^0$ and define $g = A_1(h_1) + A_2(h_2)$.

Then $g \in H^\infty(X^0)$, $\|g\| \leqq 2c_4M\|f\|$ and $\|f - g\|_S = \|A_1(h_1) - h_1 + A_2(h_2) - h_2\|_S \leqq \varepsilon M\|f\| \leqq 1/2\|f\|$ if we choose $\varepsilon \leqq 1/2M$.

Put $g_1 = g$ and assume $g_1, \cdots, g_n$ constructed such that

$$\|g_k\| \leqq 2^{-k+2}c_4M\|f\| \text{ for } 1 \leqq k \leqq n$$

and

$$\left\|f - \sum_1^n g_j\right\|_S \leqq \frac{\|f\|}{2^n}\,.$$

By the approximation technique above one easily find $g_{n+1} \in H^\infty(X^0)$ such that $\|g_{n+1}\| \leqq 2^{-n-1}c_4M\|f\|$ and

$$\left\| f - \sum_1^{n+1} g_j \right\|_S \leqq \frac{\|f\|}{2^{n+1}} .$$

By induction the series $\sum_1^\infty g_n \in H^\infty(X^0)$ interpolates f on S.

LEMMA 2. *Let S be a sequence in X^0 with no clusterpoints in X^0. Assume there exist n points $z_1, \cdots, z_n$ and numbers $r_k > s_k > t_k$ for $1 \leqq k \leqq n$ such that the open discs $\{\Delta(z_k, t_k)\}_{k=1}^n$ cover $\bar{S}$.*

Assume also that $(C\backslash X) \cap \{w: r_k > |w - z_k| > s_k\}$ and $(C\backslash X) \cap \{w: |s_k > |w - z_k| > t_k\}$ are nonempty for each k. If for each k, $\Delta(z_k, r_k) \cap S$ is an interpolating sequence for $H^\infty(X^0)$ then also S is.

Proof. We can assume $n \geqq 2$ and by induction the lemma to be true if n is replaced by $n - 1$.

Put $S_1 = S \cap \Delta(z_n, t_n)$.

By hypothesis $S_2 = S \cap (\bigcup_1^{n-1} \Delta(z_k, s_k))$ is an interpolating sequence for $H^\infty(X^0)$ and given $f \in C(S)$ we can find $h_1 \in H^\infty(X^0)$ such that $h_1 = f$ on S_2.

The choose $h_2 \in H(X^0)$ equal to $f - h_1$ on $\Delta(z_n, r_n)$.

By Lemma 1 we can find h_3 in $H^\infty(X^0)$ such that $h_3 = 1$ on S_1 and $h_3 = 0$ on $S_2\backslash\Delta(z_n, s_n)$.

Then $h_1 + h_2 h_3 = f$ on S.

Proof of Theorem 1. We have to show that the local condition implies that S is an interpolating sequence.

S has no clusterpoints in X^0 and for each $z \in (\partial X) \cap \bar{S}$ we can find $r_z > 0$ and such that $\Delta(z, r_z) \cap S$ is an interpolating sequence for $H^\infty(X_z^0)$ where $X_z^0 = \{w: |w - z| < 2r_z\} \cap X^0$. By Lemma 1 $S \cap \Delta(z, r_z)$ is an interpolating sequence for $H^\infty(X^0)$.

Since $z \in \partial X$ we can choose $s_z > t_z > 0$ such that $(C\backslash X) \cap \{w: r_z > |w - z| > s_z\}$ and $(C\backslash X) \cap \{w: s_z > |w - z| > t_z\}$ are nonempty.

Since $\bar{S} \cap (\partial X)$ is compact we can obtain the hypothesis of Lemma 2 for a set $S' \subset S$ such that $S\backslash S'$ is finite.

But if S' is an interpolating sequence for $H^\infty(X^0)$ then clearly also S is.

REMARK. To prove Theorem 1 in case $H = HR(X)$ one must modify the arguments slightly in the proof of Lemma 1. We use the notation from that lemma.

Given $f \in C(S)$ one finds $h_i \in HR(\bar{0}_i \cap X)$ equal to f on S_i such that $\|h_i\|_{HR} \leqq M\|f\|$ where M is a constant independent of S found by using the open mapping theorem.

Then we find a sequence $\{g_n^i\}_{n=1}^\infty \subset C(S^2)$ analytic in a neighbourhood of $X \cap \overline{0}_i$ (depending on n) such that $\sup_n ||g_n^i|| \leqq 2M||f||$ and such that $g_n^i \rightarrow h_i$ pointwise on the interior of $X \cap \overline{0}_i$. (S^2 denotes the extend complex plane with the usual topology.) We can also assume g_n^i converges in the w^*-topology of $L^\infty(dxdy)$ to a function $\widetilde{h}_i$ equal to h_i on $0_i \cap X^0$ such that $||\widetilde{h}_i||_\infty \leqq 2M||f||$.

We can assume $\widetilde{h}_i = 0$ outside $\overline{0}_i$.

Then it is easy to see that $\sum_{i=1}^2 A_i(\widetilde{h}_i)$ will approximate f well on S and that $A_i(g_n^i)|X$ belongs to $R(X)$ for all n and that $A_i(g_n^i) \rightarrow A_i(\widetilde{h}_i)$ pointwise on X^0. Also $||A_i(\widetilde{h}_i)||_{HR} \leqq k \cdot M||f||$. ($k$ is independent of f.)

With these remarks Lemma 1 also applies for $HR(X)$. It is clear that similar modiffications give Lemma 1 also for $HA(X)$.

But then the rest of the proof of Theorem 1 including the proof of Lemma 2 applies almost directly.

COROLLARY 1. *Let X be a compact plane set and E_a closed subset.*

Then E is an interpolation set for $R(X)$ if and only if for each $z \in E$ there exists a closed disc $N_z = \{w: |w - z| \leqq r_z\}$ such that $E \cap N_z$ is an interpolation set for $R(X \cap N_z)$.

Proof. Clearly $E_z = E \cap \{w: |w - z| \leqq z/2\}$ is an interpolation set for $R(X \cap N_z)$.

The approximation technique used in the proof of Lemma 1 shows that E_z then is an interpolation for $R(X)$.

But then the corollary follows from Rainwaters result.

REMARK. A similar corollary also clearly holds for $A(X)$.

Finally we state a theorem for $R(X)$ which is not difficult to prove. Perhaps it makes the space $HR(X)$ a little more attractive.

THEOREM 3. *Let S be a closed subset of a compact plane set X. Suppose that*

(i) *$S \cap \partial X$ is a peak interpolation set for $R(X)$*

(ii) *$S \cap X^0$ is an interpolating sequence for $HR(X)$.*

Then $R(X)|S = C(S)$.

One proves Theorem 3 by showing that for every $f \in C(S)$ there exists $g \in R(X)$ such that

$$||f - g||_S \leqq \frac{1}{2}||f|| \text{ and } ||g|| \leqq M||f||$$

where M is independent of f. This is sufficient by the approximation argument at the end of the proof of Lemma 1.

First choose $f_1 \in R(X)$ such that $f_1 = f$ on $S \cap \partial K$ and $||f_1|| \leqq ||f||$.

Interpolate then $f - f_1$ on $S \cap X^0$ by $f_2 \in HR(X)$ such that $||f_2||_{HR} \leqq M_1||f||$ where M_1 is independent of f.

If $\varepsilon > 0$ choose an open set $V_\varepsilon \supset S \cap \partial X$ such that $|f_2| < \varepsilon$ on $S \cap X^\circ \cap V_\varepsilon$.

Choose also $f_3 \in R(X)$ such that $||f_3| \leqq 2$, $f_3 = 0$ on $S \cap \partial X$ and $|1 - f_3| < \varepsilon$ on $X \backslash V_\varepsilon$ and $f_4 \in R(X)$ such that $|f_4(z) - f_2(z)| \leqq \varepsilon$ for all $z \in S$ where $|f_3(z)| \geqq \varepsilon$ and such that $||f_4|| \leqq 2||f_2||_{HR} \leqq 2M_1||f||$.

Then put $g = f_1 + f_3 f_4$. We have $||g|| \leqq (1 + 4M_1)||f||$ and $||f - g||_s \leqq \varepsilon ||f|| (2 + 3M_1)$.

So with $\varepsilon = 1/(4 + 6M_1)$ we have what we want.

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UNIVERSITY OF OSLO