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PLANAR IMAGES OF DECOMPOSABLE CONTINUA

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A nondegenerate metric space that is both compact and connected is called a continuum. In this paper it is proved that if M is a continuum with the property that for each indecomposable subcontinuum H of M there is a continuum K in M containing H such that K is connected im kleinen at some point of H and if f is a continuous function on M into the plane, then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable. Consequently, if M is a continuum in Euclidean n-space that does not contain an indecomposable continuum in its boundary, then no planar continuous image of M has an indecomposable continuum in the boundary of one of its complementary domains.

For a given set Z, the closure and the boundary of Z are denoted by Cl Z and Bd Z respectively. The union of the elements of Z is denoted by St Z.

THEOREM 1. *If X is a continuum in a 2-sphere S and I is an indecomposable subcontinuum of X that is contained in the boundary of a complementary domain of X, then every subcontinuum of X that contains a nonempty open subset of I contains I.*

Proof. Let D be a complementary domain of X such that $I \subset \text{Bd } D$, and let $X' = S - D$. By Theorem 1 of [1], every subcontinuum of X', and hence every subcontinuum of X, which contains a nonempty open subset of I contains I.

DEFINITION. An indecomposable subcontinuum I of a continuum X is said to be *terminal* in X if there exists a composant C of I such that each subcontinuum of X that meets both C and $X - I$ contains I.

THEOREM 2. *Suppose X is a plane continuum, I is an indecomposable subcontinuum of X, and each subcontinuum of X that contains a nonempty open subset of I contains I. Then I is terminal in X.*

Proof. Suppose there exists a collection E of continua in X such that for each composant C of I there is an element of E that meets both C and $X - I$ and does not contain I. Let $\{U_n\}$ be the elements of a countable base (for the topology on the plane) that intersect I. For each positive integer n, let P_n be the set consisting of all components Q of $I - U_n$ such that Q meets an element of E that is contained in $X - \mathrm{Cl}\, U_n$. Since $I = \bigcup_{n=1}^{\infty} \mathrm{St}\, P_n$, for some integer n, the set $\mathrm{St}\, P_n$ is a second category subset of I. Let L be the set consisting of all elements B of P_n such that there exists a subcontinuum F of an element of E contained in $X -\mathrm{Cl}\, U_n$ with the property that F meets both B and $X - I$ and does not intersect $I - B$. According to a theorem of Kuratowski's [3], $\mathrm{St}\, L$ is a first category subset of I. Let J denote the set of all elements H of E such that H is contained in $X - \mathrm{Cl}\, U_n$ and meets an element of $P_n - L$. Define R to be the union of all components of $\mathrm{St}\,(J \cup P_n)$ that intersect the set $\mathrm{St}\, J$. Each element of J meets three elements of P_n. Hence each component of R contains a triod. It follows that the components of R are countable. Since $\mathrm{St}\,(P_n - L)$ is a second category subset of I that is contained in R, there exists a component T of R such that $\mathrm{Cl}\, T$ contains a nonempty open subset of I. But since $\mathrm{Cl}\, T$ is a continuum in $X - U_n$, this is a contradiction. Hence I is terminal in X.

THEOREM 3. *Suppose M is a continuum with the property that for each indecomposable subcontinuum H of M there is a continuum K in M containing H such that K is connected im kleinen at some point of H and f is a continuous function on M into the plane. Then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.*

Proof. Suppose a complementary domain of $f(M)$ contains an indecomposable continuum I in its boundary. According to Theorems 1 and 2, I is terminal in $f(M)$. Hence there exists a composant C of I such that each subcontinuum of $f(M)$ that meets both C and $f(M) - I$ contains I. Let p be a point of $f^{-1}(C)$. Define Z to be the p-component of $f^{-1}(I)$. As in the proof of Theorem 2 of [2], $f(Z) = I$.

Let A be a composant of I distinct from C. There exists a continuum H in Z such that $f(H)$ meets A and C, and no proper subcontinuum of H has an image under f that meets both A and C. Note that $f(H) = I$ and H is indecomposable. There is a continuum K in M containing H that is connected im kleinen at some point of H. Hence there exists a continuum W in K whose interior (relative to K) meets H such that $f(W)$ does not contain I. Each composant of H meets W.

Let x be a point of $H \cap f^{-1}(C)$. Since the x-composant of H intersects W, it follows that $f(W)$ is contained in C. Let y be a point of $H \cap f^{-1}(A)$. There exists a proper subcontinuum Y of H that contains y and meets W. Since $f(Y)$ meets both A and C, this is a contradiction. Hence the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.

COROLLARY 1. *If a continuous image of a hereditarily decomposable continuum lies in the plane, then the boundary of each of its complementary domains is hereditarily decomposable.*

COROLLARY 2. *If M is a continuum in Euclidean n-space that does not contain an indecomposable continuum in its boundary and f is a continuous function on M into the plane, then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.*

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